MASSACHUSETTS INSTITUTE OF TECHNOLOGY
ARTIFICIAL INTELLIGENCE LABORATORY

A. I. Memo 861 **September, 1985**

The Synthesis of Force-Closure Grasps

In the Plane

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Abstract: This paper addresses the problem of synthesizing planar grasps that have force closure. A grasp on an object is a force closure grasp if and only if we can exert, through the set of contacts, arbitrary force and moment on this object. Equivalently, any motion of the object is resisted by a contact force, that is the object cannot break contact with the finger tips without some non-zero external work.

The force closure constraint is addressed from three different points of view: mathematics, physics, and computational geometry. The last formulation results in fast and simple polynomial time algorithms for directly constructing force closure grasps. We can also find grasps where each finger has an independent region of contact on the set of edges.

Keywords: Planar grasps, force closure, grasp planning or synthesis.

Acknowledgments: This report describes research done at the Artificial Intelligence Laboratory of the Massachusetts Institute of Technology. Support for the Laboratory's Artificial Intelligence research is provided in part by a grant from the System Development Foundation, and in part by the Advanced Research Projects Agency under Office of Naval Research contracts N00014-80-C-0505 and N00014-82-K-0334.

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1 Introduction

1.1 Planar Force-Closure Grasps

Robot end effectors have evolved from simple parallel grippers to multi-purpose hands to provide greater flexibility and dexterity in manipulation and assembly operations. Robot hands come in many shapes, but they all have in common the ability of being programmed and servoed from a computer. To take full advantage of the dexterity offered by multi-purpose hands, we need to be able to not only *analyze* a grasp but *synthesize* it. In other words, we would like to *plan* grasps that have such features as force closure, feasibility, reachability, compliance, equilibrium, stability, etc...

This paper addresses the problem of synthesizing planar grasps that have force closure. A grasp on an object B is a force closure grasp if and only if we can exert arbitrary force and moment on object B by pressing the finger tips against this object. Equivalently, any arbitrary motion of object B will be resisted by a contact force from the fingers, which means that B cannot break contact with the finger tips without some non zero external work. That is, the total freedom of B is zero.

Inputs to the synthesis process will be the shape of the grasped object and the available set of contacts. Contacts between the object and the hand and fingers can be point contacts, edge contacts with/without friction, or soft contacts. Output will be how many required contacts, what type of contact, and where to contact on the object. The set of contacts found describes a force closure grasp of the object.

Force closure is only one necessary condition for grasp synthesis. A number of other conditions may be required. For example, we must fit a specific hand and fingers on the set of contacts found, and check for the geometric feasibility of the grasp. Next, we must plan motions of the hand and fingers and check for reachability. These motions can be position-controlled motions, or compliant motions with the fingers modeled as springs or dampers. Once the object is grasped, it can be lifted or kept in equilibrium by exerting necessary contact forces on the object. By definition, a force closure grasp guarantees that any force and moment can be decomposed into a positive combination of contact forces. So, with a suitable control loop, not only can equilibrium be maintained, but the object can also be manipulated between the fingers. In general, the set of force closure grasps is infinite. We may want to search for locally stable grasps, or the most stable grasp, which correspond to minima of the potential function of the hand and fingers.

In this paper, we restrict ourselves to the core problem of planning force closure grasps in terms of contacts between the grasped object and the hand. We feel that force closure is the most basic constraint in grasping an object, because it captures the basic scenario of many bodies contacting and constraining one another. Force closure is also the simplest constraint and is suitable for planning. The current

synthesis deals only with planar objects. All forces exerted through the contacts will lie in the plane of the object, and all moments will have axes perpendicular to this plane. The contacts and grasps are said to be planar to differentiate them from the more general contacts and grasps on 3D objects.

1.2 Contributions

The main contributions of this paper are:

• A thorough understanding of the force closure constraint in the domain of planar grasps. — Force closure is addressed from three different points of view:

∘ A mathematics point of view which casts the force closure problem as one of solving a system of linear inequations. The spaces of wrenches and twists correspond to the dual spaces of rows and columns of this system of linear inequations.

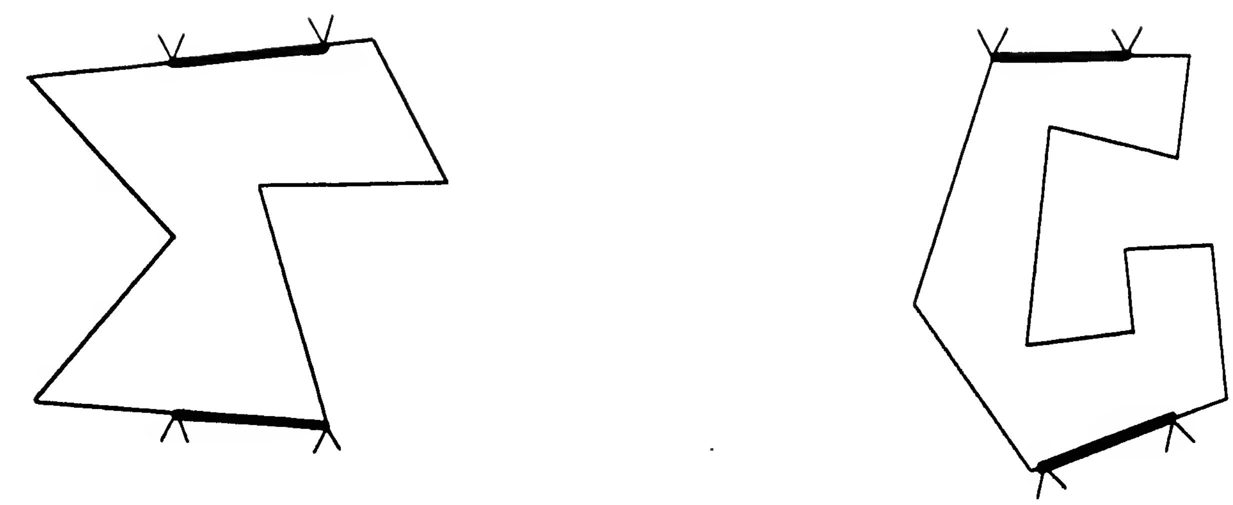
∘ A physics point of view which exploits features of planar mechanics and decomposes the force closure problem in the plane into two independent subproblems: one of force-direction closure or no free translation, and one of torque closure or no free rotation.

∘ A computational geometry point of view which results in a direct construction of force closure grasps.

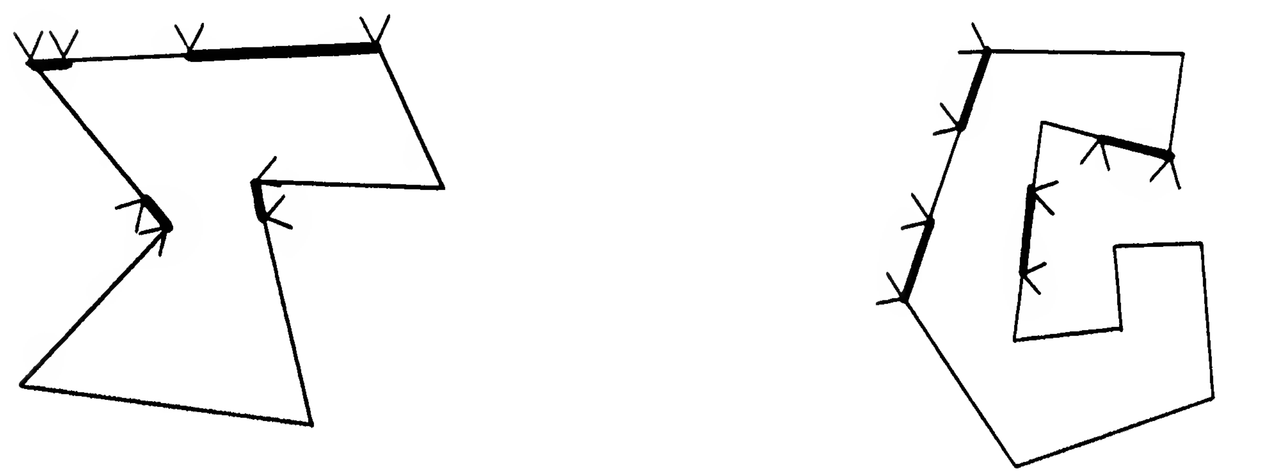
• Fast and simple algorithms for directly constructing force closure grasps. — We find not only single grasps but the complete set of all force closure grasps on a set of edges. Within this complete set, we also find the maximal set of grasps where each finger can be positioned arbitrarily in a range of contact. Finding a grasp or a set of grasps on a set of edges costs constant time. Enumerating all sets of grasps on the object is polynomial in the number of edges of the object.

• A representational framework for describing contacts and grasps. — A planar grasp is described as the combination of individual planar contacts, which in turn are modeled as the combination of a few primitive contacts. Grasps and contacts are mathematically described from two dual view points: a constraint view point which captures the forces and moments exerted on the object, and a freedom view point which describes the motions of the object which are free or which break contact. We also develop an explicit representation for describing a set of grasps on a set of edges.

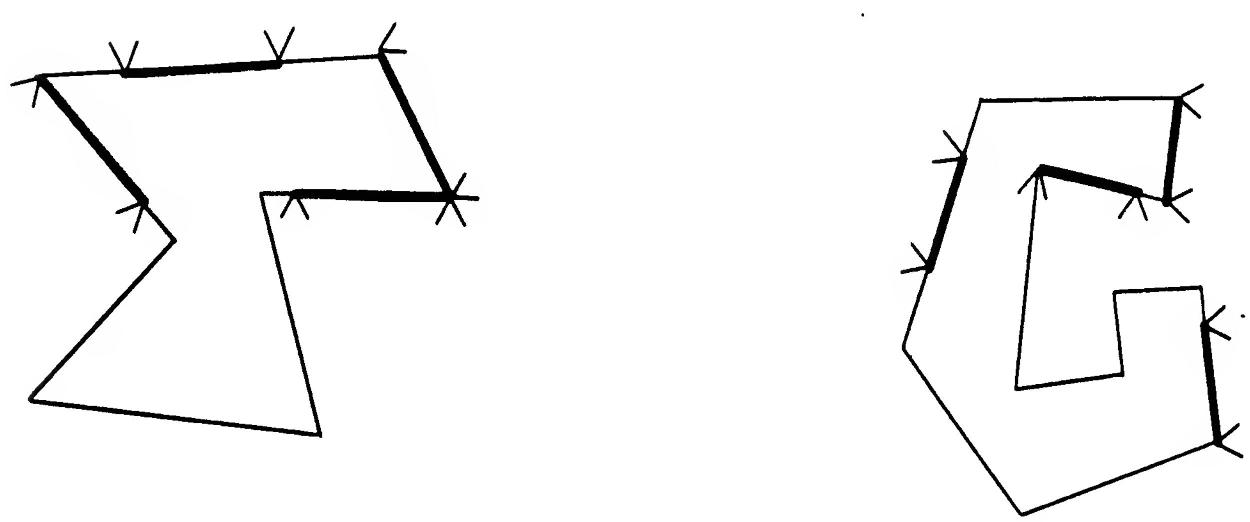
• A partial explanation to why people tend to grasp objects at sharp edges and vertices, and to why the contacting surfaces of our fingers had better be soft than hard like the finger nails. And an answer to the question: "How should robot hands grasp this object?".



a. Grasps with friction on two edges.



b. Frictionless grasps on three edges.



c. Frictionless grasps on four edges.

Figure 1: Examples of grasps found by the synthesis.
The direction of contact of the fingers is shown by small V's. The independent regions of contact are highlighted with bold segments.

1.3 Examples

Figure 1 shows samples of grasps found by the algorithms. The direction of contact of the fingers are shown by small V's. The independent regions of contact are highlighted with bold segments. No matter where the fingers are positioned in these regions, the grasp always has force closure. This flexibility is of great importance in manipulation since we always have positioning errors and many other uncertainties.

A grasp with two point contacts requires friction between the fingers and the grasped object. Without friction, we need at least four contacts if these are point contacts. We'll see that edge contacts and soft finger contacts can be viewed as a combination of many point contacts. This is why the algorithms only need to output grasps in terms of point contacts with/without friction.

1.4 Other Work

There has been extensive work on designing and controlling dextrous hands (Asada 1979, Salisbury 1982, Jacobsen et al. 1984). Robot hands come in many varieties: human-like hands like the Utah/MIT hand (Jacobsen et al. 1984), the Okada hand (Okada 1982), hands with symmetric fingers like the Asada hand (Asada 1979), the Salisbury hand (Salisbury 1982). The fingers of these hands have either position control (Okada 1982), velocity control (Salisbury 1984), or force control (Jacobsen et al. 1984) at the innermost loop, and other control modes at outer loops such as stiffness control (Salisbury 1980, Salisbury and Craig 1981).

Degrees of freedom (Hunt 1978), total freedom and force closure of mechanisms (Ohwovoriole 1980) have been fully investigated. Ohwovoriole analyzed the geometry of the different repelling screw systems, and use the results to analyze systems of contacting bodies such as an object grasped by a set of fingers, or a pin being inserted into a hole (Ohwovoriole 1980, 1984). Screws, twists, and wrenches (Bottema and Roth 1979), spatial vectors (Featherstone 1984) are not new, but strong interest within robotics has surfaced only recently.

There are many papers on analyzing grasps ranging from classification based on how people and animal grasp objects (Lyons 1985), computing the necessary forces and moments for equilibrium (Abel et. al 1985, Holzmann and McCarthy 1985), achieving stable grasps (Hanafusa and Asada 1977, Baker et. al 1985, Nguyen 1985), to modeling and analyzing the grasping operations (Mason 1982, Cutkosky 1984, Kerr 1984, Fearring 1984, Jameson 1985). There are works on analyzing force closure grasps (Ohwovoriole 1980, 1984), and on solving systems of linear inequations (Hunt and Tucker editors 1956, Strang 1976). Unfortunately, there has not been any work on directly constructing or synthesizing force closure grasps.

2 Background Theory of Twists and Wrenches

The instantaneous motion of an object is described by a twist. A twist is a spatial vector which captures both the angular and linear displacement of the object. We use wrenches to describe the system of forces and moments exerted on the object. Twists and wrenches are represented in Plücker coordinates.

We review Plücker line coordinates, virtual work and total freedom. Then we look at the dual systems of twists and wrenches in the plane. For more extensive materials on these topics, the reader is referred to (Featherstone 1984).

2.1 Plücker Line Coordinates

A general spatial vector is the sum of a line vector and a free vector. A line vector has a magnitude and a line of action, where as a free vector has magnitude and direction only. In rigid body dynamics, line vectors represent quantities like force, angular velocity which have a definite line of action. The line of action is respectively the line of force and the axis of rotation. Free vectors describe quantities like torque and linear velocity which do not change under translation.

A free vector is represented by a spatial vector with zero upper half. For example, a linear velocity $\mathbf{v}$ is represented in Plücker coordinates by the following twist:

$$\mathbf{t} = \begin{bmatrix} \mathbf{0} \\ \mathbf{v} \end{bmatrix}$$

A line vector $\mathbf{u}$ is represented by a spatial vector $\mathbf{s}$ with six Plücker coordinates. The first three coordinates represent the magnitude and direction of the vector $\mathbf{u}$ at the origin of the reference frame. The later three represent the moment of the vector $\mathbf{u}$ about the origin of the reference frame. Concretely, a line vector $\mathbf{u}$ is represented as:

$$\begin{aligned} \mathbf{s} &= \begin{bmatrix} \mathbf{u} \\ \mathbf{r} \times \mathbf{u} \end{bmatrix} \\ &= \begin{bmatrix} u_x \\ u_y \\ u_z \\ r_y u_z - r_z u_y \\ r_z u_x - r_x u_z \\ r_x u_y - r_y u_x \end{bmatrix} \end{aligned} \tag{1}$$

where $\mathbf{r}$ is a vector from the origin of the reference frame to any point on the line of action of $\mathbf{u}$.

For example, a rotation with angular velocity ω is represented by the following twist velocity in Plücker coordinates:

$$\mathbf{t} = \begin{bmatrix} \omega \\ \mathbf{r} \times \omega \end{bmatrix} \tag{2}$$

where $\mathbf{r}$ is a vector from the origin of the reference frame to any point on the axis of rotation. We recognize the upper and lower halves of the twist velocity as the angular and linear velocities of the origin of the reference frame due to rotation $\boldsymbol{\omega}$.

Like angular velocity, a force $\mathbf{f}$ is a line vector which is represented in Plucker coordinates by the following wrench:

$$\mathbf{w} = \begin{bmatrix} \mathbf{f} \\ \mathbf{r} \times \mathbf{f} \end{bmatrix} \tag{3}$$

Force $\mathbf{f}$ can be decomposed into a sum of an equivalent force going through the origin of the reference frame and a pure torque which is the moment of the force $\mathbf{f}$ about the origin. Equivalently, wrench $\mathbf{w}$ can be written as the sum of a line vector and a free vector:

$$\mathbf{w} = \begin{bmatrix} \mathbf{f} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{r} \times \mathbf{f} \end{bmatrix}$$

We note that force $\mathbf{f}$ has a different wrench when its line of action is translated to the origin. By representing the contact forces and the instantaneous motion of the grasped object in Plücker coordinates, we make explicit the line-based geometry of the domain.

2.2 Virtual Work and Total Freedom

Twists and wrenches are related by a spatial scalar product which describes the virtual work done by the wrench against the twist. The virtual work is defined as follows:

Definition 2.1 *The virtual work of a wrench* $\mathbf{w} = [\mathbf{f} \mid \mathbf{m}]$ *against an infinitesimal twist* $\mathbf{t} = [\mathbf{d}\boldsymbol{\alpha} \mid \mathbf{dx}]$ *is the sum of the virtual work due separately to the linear and angular components:*

$$\mathbf{w} \,\hat{\cdot}\, \mathbf{t} = \mathbf{f} \cdot \mathbf{dx} + \mathbf{m} \cdot \mathbf{d}\boldsymbol{\alpha} \tag{4}$$

We can define the virtual work as a scalar multiplication of the wrench $\mathbf{w}$ *with the spatial transpose of the twist* $\mathbf{t}$*, denoted* $\mathbf{t}^s$*:*

$$\begin{aligned} \mathbf{w} \,\hat{\cdot}\, \mathbf{t} &= \mathbf{w} \cdot \mathbf{t}^s \\ &= \begin{bmatrix} \mathbf{f} \\ \mathbf{m} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{dx} \\ \mathbf{d}\boldsymbol{\alpha} \end{bmatrix} \end{aligned}$$

The spatial transpose of twist $\mathbf{t} = [\mathbf{d}\boldsymbol{\alpha}|\mathbf{dx}]$ *is the twist* $\mathbf{t}^s = [\mathbf{dx}|\mathbf{d}\boldsymbol{\alpha}]$ *with the upper and lower halves permuted.*

The concept of total freedom (Ohwovoriole 1980) emerges from the sign of the virtual work. We'd like to know not only the degrees of freedom of the object under a system of wrenches and twists, but also whether the object breaks contact or pushes against the other bodies contacting it.

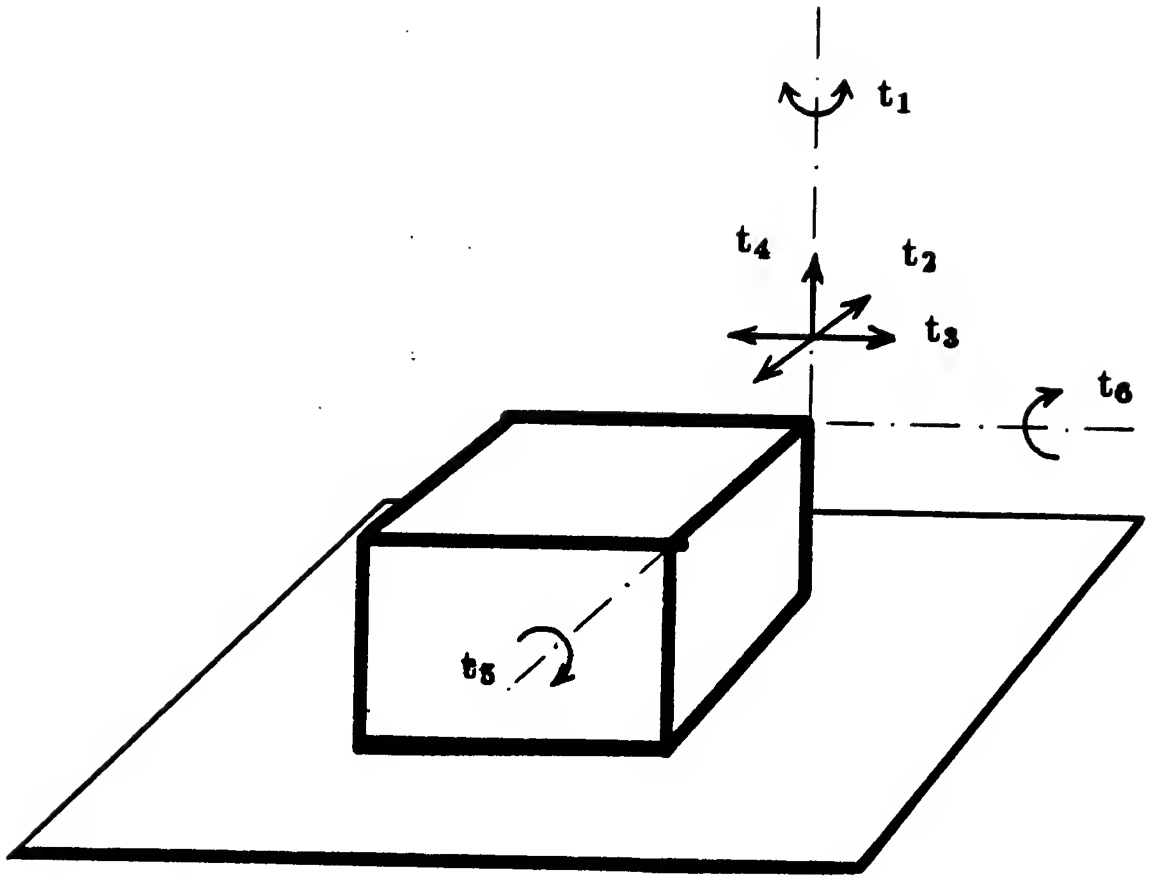


Figure 2: Total freedom of a box lying on a frictionless plane.

Definition 2.2 *A twist* t *and a wrench* w *are* reciprocal *to each other if and only if their virtual work is zero. The pair* (t, w) *is* repelling, *(resp.* contrary*), if and only if their virtual work is strictly positive, (resp. negative).*

An object constraining by a set of wrenches W has n degrees of freedom *if and only if the set of twists reciprocal to W can be represented as* linear combinations *of n linearly independent twists.*

Similarly, the object has n total freedoms *if and only if the set of twists reciprocal and repelling to W can be generated from* non-negative combinations *of n non-zero twists. These n twists form a minimal generating basis.*

A reciprocal twist corresponds to a degree of freedom in the system. For example, let's look at the box in Figure 2. The box lies on a frictionless horizontal plane. It has a reciprocal rotational twist about any vertical axis. In other words, it can freely rotate about any vertical axis, and therefore has one degree of freedom. The box can also translate in the plane, and so has two other degrees of freedom.

The above three degrees of freedom do not completely describe the set of motions possible to the object. For example, the box can be raised from the plane and break contact. This half-free motion is different from the usual concept of degree of freedom in that it is *unisense.* The upward motion is a twist repelling to the contact force exerted by the plane onto the box. The downward motion is a twist contrary to this contact force. Note that the plane will oppose the downward motion with

an upward reaction force, resulting in a negative virtual work. Finally, the box can rotate about two horizontal axes, provided that the box does not enter the plane. These two rotations define two other total freedoms. The box has three degrees of freedom, but its total freedom is really six.

2.3 Convexes and Operations on Convexes

In grasping, the goal is to have force closure or to fully constrain the motion of the grasped object with a set of finger contacts. Through each finger contact we can exert a range of forces and moments, which can be represented by a wrench convexe. Just as many contacts are combined to form a grasp, many wrench convexes are 'added' until their sum spans the whole space, or until we have force closure. Each finger contact can also be described by a twist convexe. The twist convexes are intersected until we get the null space, or until the grasped object has zero total freedom. Let's first define convexes and three operations on convexes: convexe addition, intersection, and dual.

Definition 2.3 *Let C be a non-empty set of vectors. We define by convexe $C^<$ the set of all non-negative combinations of vectors in C, formally:*

$$C^< = \left\{ \mathbf{s} \mid \mathbf{s} = \sum_i \alpha_i \mathbf{s}_i,\ \alpha_i \geq 0,\ \mathbf{s}_i \in C \right\} \tag{5}$$

The vectors $\mathbf{s}_i$ in C are called generating vectors of the convexe $C^<$.

A convexe is null *when it contains only the null vector. A convexe is* total *when it is the entire space.*

An example of a convexe is the friction cone in the plane, Figure 3. The range of forces inside the friction cone can be represented as the set of all positive combinations of the two extreme rays of the friction cone. So a friction cone at point P with normal $\mathbf{n}$ and friction angle ϕ can be represented by a two-wrench convexe:

$$W^< = \left\{ \begin{bmatrix} rot(-\mathbf{n}, \phi) \\ \mathbf{p} \times rot(-\mathbf{n}, \phi) \end{bmatrix}, \begin{bmatrix} rot(-\mathbf{n}, -\phi) \\ \mathbf{p} \times rot(-\mathbf{n}, -\phi) \end{bmatrix} \right\}$$

where $rot(\mathbf{x}, \theta)$ is the direction vector $\mathbf{x}$ rotated by θ.

Definition 2.4 *Let $C_1^<$, $C_2^<$ be two convexes. The convexe addition of $C_1^<$ and $C_2^<$, denoted $C_1^< \cup C_2^<$, is the least convexe that contains both $C_1^<$ and $C_2^<$. Formally:*

$$\begin{aligned} C_1^< \cup C_2^< = \{ \mathbf{s} \mid & \ \mathbf{s} = \alpha \mathbf{s}_1 + \beta \mathbf{s}_2, \\ & \alpha \geq 0,\ \beta \geq 0,\ \mathbf{s}_1 \in C_1^<,\ \mathbf{s}_2 \in C_2^< \} \end{aligned} \tag{6}$$

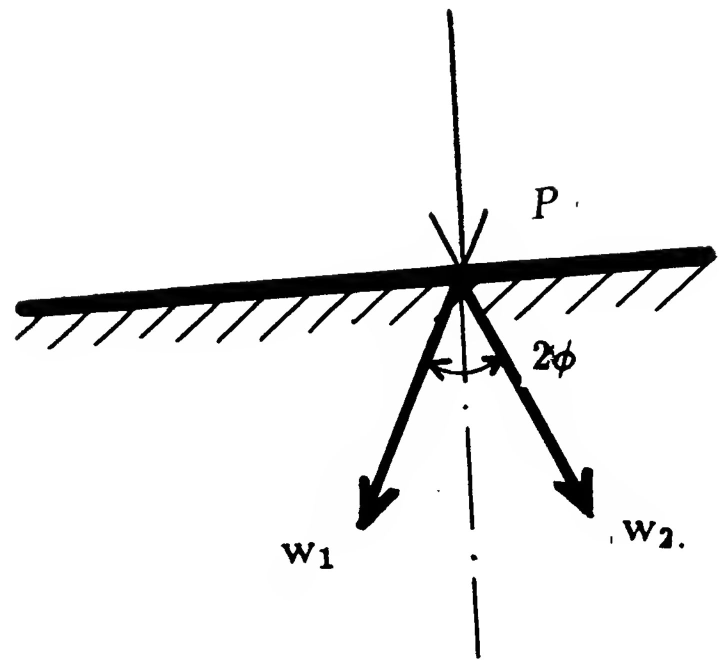


Figure 3: A friction cone is represented by a two-wrench convexe.

Convexe addition [1] is also known as the Minkowski's sum (Najfeld et al. 1980). An example of convexe addition is the combination of wrench convexes from more than one contact. For example, each point contact with friction gives a two-wrench convexe. A grasp with two point contacts with friction has a wrench convexe which is the Minkowski sum of the two convexes, describing the friction cone at each point contact. From the above equation, the resulting convexe can be generated by the four wrenches, each is a force along an edge of the two friction cones. We note that representing the convexe sum is trivial, except that the set of generating wrenches can be redundant.

Convexes are closed under convexe addition and intersection. The intersection of two convexes is defined as follows:

Definition 2.5 *Let $C_1^<$, $C_2^<$ be two convexes. The intersection of $C_1^<$ and $C_2^<$, denoted $C_1^< \cap C_2^<$, is the largest convexe inside both $C_1^<$ and $C_2^<$. Formally:*

$$C_1^< \bigcap C_2^< = \left\{ \mathbf{w} \mid \mathbf{w} \in C_1^<, \mathbf{w} \in C_2^< \right\} \tag{7}$$

Twist and wrench convexes are duals of each other. The dual operation on twist or wrench convexes is defined as follows:

Definition 2.6 *Let $C^<$ be a twist or wrench convexe. The dual of $C^<$, denoted $\overline{C^<}$, is the convexe of vectors that are either reciprocal or repelling to all the vectors*

[1] Convexe addition is *not* set union. We borrow the union sign $\bigcup$ to emphasize the duality of the convexe addition and the set intersection of two convexes, denoted $\bigcap$.

in $C^{<}$. *Formally:*

$$\overline{C^{<}} = \{ \mathbf{s} \mid [C]\, \mathbf{s}^s \geq \mathbf{0} \} \tag{8}$$

where $[C]$ *is the matrix whose rows are generating vectors of the convexe* $C^{<}$.

One use of the dual operation is to calculate the twist convexe $T^{<}$ describing the total freedom of the system from the wrench convexe $W^{<}$. Solving for the twist convexe is equivalent to solving the following system of linear inequations:

$$[W]\, \mathbf{t}^s \geq \mathbf{0}$$

where $\mathbf{t}^s$ is the spatial transpose of the unknown twist, and each row of matrix $[W]$ is a wrench of W. The product of $\mathbf{t}^s$ and the ith row of $[W]$ gives the virtual work of twist $\mathbf{t}$ against wrench $\mathbf{w}_i$ of W. This virtual work must be either zero or positive, and similarly for all other rows.

The duality between twists and wrenches allows us to compute in the wrench-space and deduce equivalent result in the twist-space, and vice versa. See Figure 5. The following Lemma summarizes important facts about the dual operation, the convexe addition and intersection of twist and wrench convexes. For a proof see (Hunt 1956).

Lemma 2.1 *Let* $W^{<}$, $T^{<}$ *be respectively a wrench and a twist convexe. Let* $C^{<}$ *be either a wrench or twist convexe.*

$$\begin{aligned}
&1.\ \overline{\overline{C^{<}}} = C^{<} \\
&2.\ \overline{W^{<}} = T^{<} \qquad \overline{T^{<}} = W^{<} \\
&3.\ C^{<} \cap \overline{C^{<}} = \textit{null space} \qquad C^{<} \cup \overline{C^{<}} = \textit{total space} \\
&4.\ \overline{C_1^{<}} = \overline{C_2^{<}} \ \textit{if and only if}\ C_1^{<} = C_2^{<} \\
&5.\ \overline{C_1^{<} \cap C_2^{<}} = \overline{C_1^{<}} \cup \overline{C_2^{<}} \\
&6.\ \overline{C_1^{<} \cup C_2^{<}} = \overline{C_1^{<}} \cap \overline{C_2^{<}}
\end{aligned} \tag{9}$$

2.4 Dual Systems

We have seen that twists and wrenches form dual systems. In planar mechanics, a twist can be represented by a 3-dimensional spatial vector as follows:

$$\mathbf{t} = \begin{bmatrix} d\alpha_z \\ dx \\ dy \end{bmatrix} \tag{10}$$

$d\alpha_z$ is the infinitesimal rotation about the z-axis, and $[dx, dy]$ is the infinitesimal translation of the origin in the xy-plane. Similarly, a planar wrench can be repre-

sented by:

$$\mathbf{w} = \begin{bmatrix} f_x \\ f_y \\ m_z \end{bmatrix} \tag{11}$$

where $[f_x, f_y]$ are the two force components in the xy-plane, and m_z is the moment component about the z-axis.

In this 3-dimensional space, we can identify two pairs of subspaces which form interesting dual systems:

- The space of xy-translations, and its dual which is the space of all force directions in the xy-plane.
- The space of pure moments or torques about the z-axis, and its dual which is the space of clockwise and counter-clockwise rotations about the z-axis.

It is well known that any planar motion can be decomposed uniquely into a translation and a rotation about the origin. So, the space clockwise and counter-clockwise rotations, and the space of xy-translations are two independent subspaces of the space of planar twists. Similarly, the space of torques and the space of force directions are two independent subspaces of the space of planar wrenches. Force closure for these two pairs is very simple, and is discussed in sections 4 and 5. For the moment, we turn away from force closure to discuss the issue of representing contact and grasp.

3 Representing Contacts and Grasps

3.1 Primitive Planar Contacts

We want to represent the range of forces and moments that can be exerted on rigid objects through a planar contact. Figure 4 depicts the different types of contact one can find in a planar grasp. The first column describes the physical contact with the finger on top and the grasped object below it. The second and third columns describe respectively the wrench convexe, representing forces that can be applied to the object, and the twist convexe, representing the total freedom of the object. Each convexe is represented by a minimal set of generating vectors. The twist convexe is computed by taking the dual of the wrench convexe.

Any planar contact can be decomposed into a combination of the following primitive contacts:

• *Frictionless point contact* — We can apply only a single pure force, normal to surface, through a frictionless point contact. The wrench convexe has a single wrench. The twist convexe has three twists: a rotation about the point of contact, a translation along the edge of the object, and an unisense downward translation

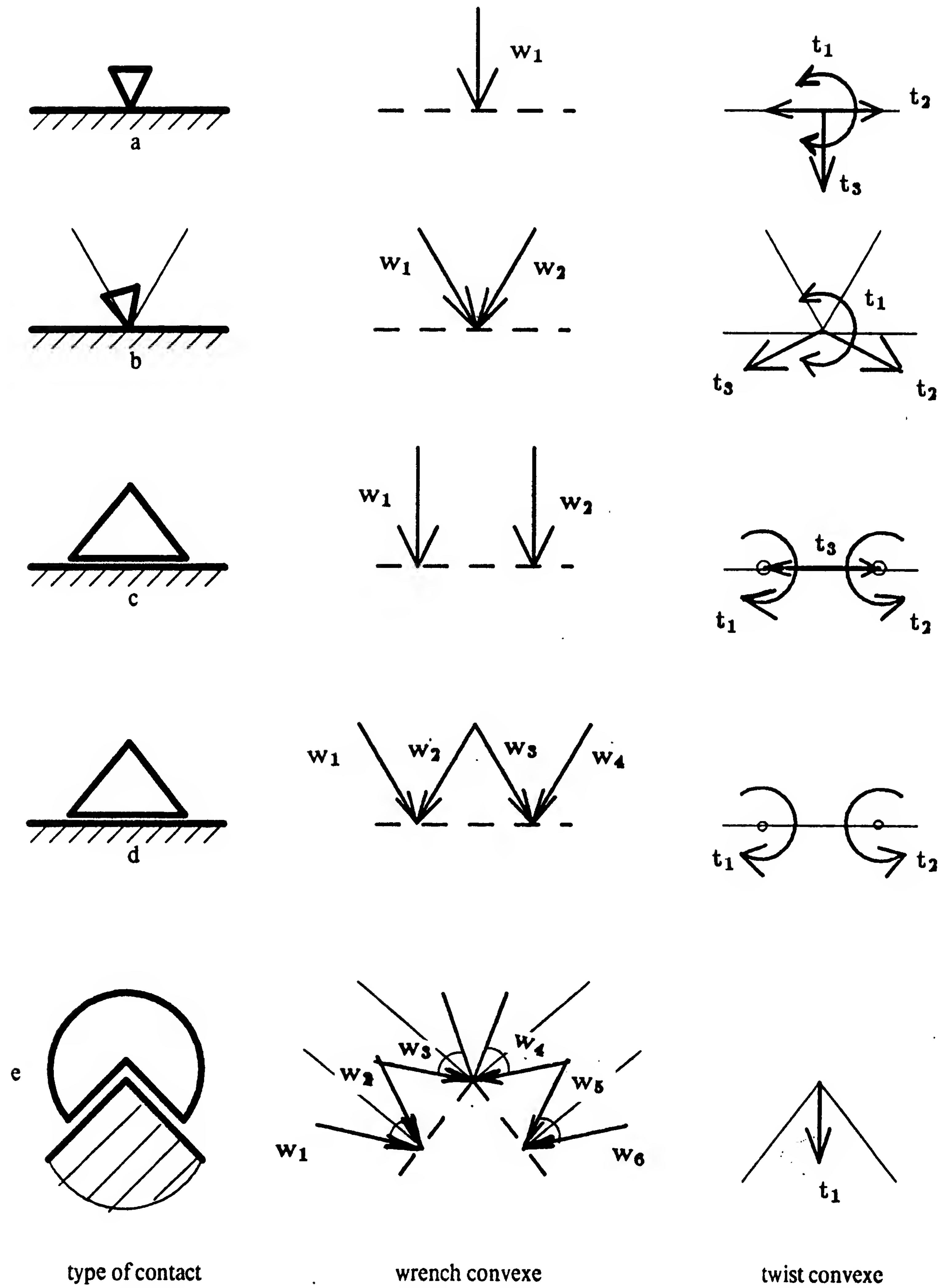


Figure 4: Primitive planar contacts and their twist and wrench convexes.

which breaks contact with the finger. This downward translation is a repelling twist where as the first two twists are reciprocal ones. Figure 4.a.

• *Point contact with friction* — The friction cone at the point of contact shows the range of pure force that can be applied through the point contact. The wrench convexe has two wrenches which are along the two extreme rays of the friction cone. Any force pointing inside this friction cone can be written uniquely as a positive combination of the these two wrenches. The twist convexe has two unisense translations, each reciprocal to one wrench and repelling to the other. It also has a free rotation about the point of contact as above. Figure 4.b.

• *Frictionless edge contact* — It is well known that for rigid objects, any force distribution along the segment of contact is equivalent to a unique force at some point inside the segment. So a frictionless edge contact gives us the ability of exerting a range of forces perpendicular to the edge and going through the segment of contact. This range of forces is mathematically the positive combination of two forces at the two ends of the segment of contact. So the wrench convexe has two wrenches.

Solving for the dual of the wrench convexe, we get a clockwise rotation about the left end point, a counter-clockwise rotation about the right end point, each breaks contact with the other end point. Since there is no friction between the two edges, the object can freely slide horizontally in both directions. Note these three twists form the minimum representative set for the twist convexe. The downward translation is not present because it can be synthesized as the sum of the two rotations about the two end points. Figure 4.c.

• *Edge contact with friction* — Instead of a single force perpendicular to the segment of contact, we can now apply any force pointing inside any friction cone inside the segment. The wrench convexe becomes the convexe addition of two convexes each representing a friction cone at one end of the segment of contact.

With friction between the two edges, the object can no longer slide horizontally without constraint. However, it can still rotate about one end of the segment of contact and break contact with the other end. Figure 4.d.

• *Soft finger contact* — From a force closure point of view, a soft finger contacting an edge is the same as an edge contact with friction. The pressure distribution is irrelevant to our domain which is concerned with whether the object can be constrained with these contacts, rather than how much force should the fingers apply to the object.

A soft finger becomes useful when it comes to contacting on the inside or outside of a corner. Figure 4.e shows a soft finger contacting on the outside of a corner. The wrench convexe is the convexe addition of two convexes, each describes the edge contact with friction on one side of the corner. The minimum number of generating wrenches is six.

The object cannot rotate about the two ends of the soft contact. But it can still break contact by sliding downward. This downward sliding is no longer repelling if

the corner angle is small enough, or if the friction cone is large enough. We'll see in subsection 6.3 that friction plays a crucial role in reducing the required number of point contacts of a planar grasp from four to two.

People tend to grasp at the edges and corners if there is no reachable pair of parallel faces. Why? One among many plausible answers is the availability of a larger wrench convexe, which means not a more stable grasp but a greater ability to constrain the object by applying necessary forces through the soft contacts. A soft contact can be approximated as a point contact with a much larger friction cone. So a grasp with two soft contacts is better than a grasp with two point contacts with friction.

Gravity is not a contact, but it does play a role in constraining the total freedom of the object. For example, the box of Figure 2 is immobile on the table because the force of gravity is holding it down to the table. We can view the box as being grasped, or more exactly constrained, by two contacts: a plane contact between the bottom of the box and the table, and an imaginary point contact at the center of gravity of the box. Gravity is a blessing in this case, because without gravity the box can freely float upwards! We can easily include the effect of gravity by imagining it as a frictionless point contact at the center of gravity of the object. A grasp with gravity may need one fewer contacts than one without gravity.

3.2 Dual Representations For Grasps

Twist and wrench convexes are two dual representations for contacts. Convexes are closed under convexe addition and intersection. We can add wrench convexes from all the contacts or intersect the corresponding twist convexes to find the resulting wrench or twist convexe of the grasp. We have here two dual view points and two equivalent ways to represent grasps:

- A constraint view point. — Wrench convexe describes the set of forces and moments which constrain the object. A total wrench convexe means we can arbitrarily apply any force and moment on the object, and so we can grasp it, instantaneously rotate or translate it in any way we want.

- A freedom view point. — Twist convexe describes the total freedom of the object. A total twist convexe means the object can freely move relative to the fingers; a null twist convexe means the object cannot break contact without external work against contact forces exerted by the fingers.

Which representation, twist or wrench convexes, is better? For planning grasps, wrench convexes are definitely more efficient [2] since generating wrenches can be

[2] We note briefly here that the twist convexe representation is more efficient for describing the total freedom at the end effectors of linked manipulators. Infinitesimal motions and velocities of the end effector due to each joint are 'added', and the end effector can have arbitrary motion if the twist convexes of all the joints 'add up' to a total convexe.

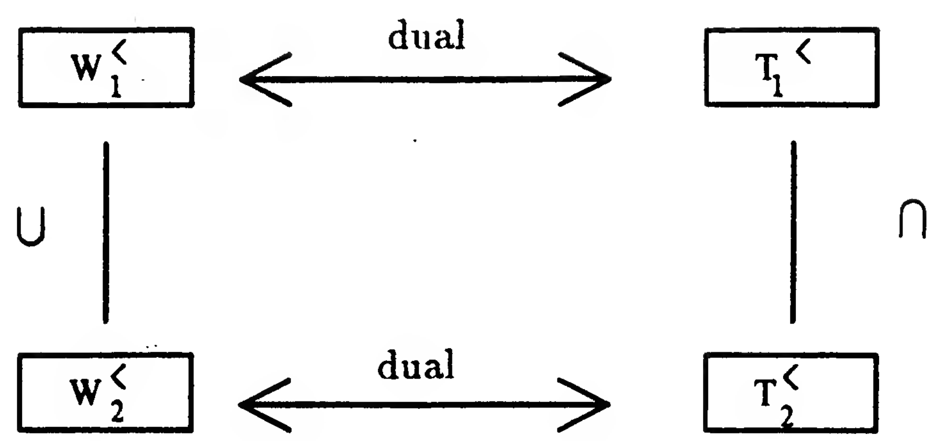


Figure 5: Duality between twist and wrench spaces

deduced readily from the type of contact, and we can just take the union of all the generating wrenches to describe the grasp. If we choose twist convexes as the representation, for each contact, we have to compute the dual of the wrench convexe to find its corresponding twist convexe, and then compute the intersection of these twist convexes, see Figure 5.

To represent the set of force closure grasps on a set of edges, we have another representation called a *grasp set*. How a grasp set is defined and calculated is presented in Section 6. Before looking at grasp sets and how they are constructed, let's step back, and ask again: "What is a force closure grasp?"

We certainly know what force closure means by now. It means the ability to exert arbitrary force and moment on the grasped object, or that the object is totally constrained. We have also seen one way of casting the force closure problem, that of solving a system of linear inequations:

$$[W]\ \mathbf{t}^s \ \geq\ 0$$

where W is the set of generating wrenches collected from all the contacts of the grasp. We can design a generate-and-test algorithm which enumerates all the possible grasps, and test each grasp by solving the above system of linear inequations. There are two main objections to this scheme: first, the set of possible grasps is infinite; second, the grasp synthesis uses an analytical formulation which blurs critical features of the domain such as the difference between a force and a pure torque.

The key is to make the force closure constraint explicit, and this is what we explore next. For a quick reading of the paper, the reader may browse through the geometrical view of force-direction and torque closure first, Sections 4.3 and 5.3. He can return for more analytical proof later. Algorithms for synthesizing force closure grasps are presented in Sections 6 and 7. Due to the explicit formulation of the force closure constraint, the algorithms are not only fast but also very simple.

4 Ability To Resist Arbitrary Translation

4.1 Force-Direction Closure with Planar Forces

When can a grasp resist arbitrary planar translation of the object? Informally, the contact forces of the grasp must have directions that span the space of all directions in the plane. An example of spanning the space of all directions in the plane is three directed rays going from the center of gravity of a triangle out to the three vertices of that triangle.

Formally, the ability of a grasp G to resist arbitrary translation of object B can be formulated in three equivalent ways shown in the following theorem:

Theorem 4.1 *Let a grasp configuration G be described by the set of wrenches W. Each wrench in W is a contact force which acts on the object B. The three following clauses are equivalent:*

1. *Any non-null translational twist $\mathbf{t} = [0 \mid v_x,\ v_y]$ of the object B is contrary to at least one wrench of W.*
2. *There is no non-null translational twist either reciprocal or repelling to all wrenches in W.*
3. *The positive combination of wrenches in W can generate force in any arbitrary direction. The grasp G is said to have force-direction closure.*

Proof: The second clause is the double negation of the first clause, so they are equivalent. They express two equivalent views: one is the existence of contact forces which resist arbitrary translation, the other is the non existence of free translation, or translation that breaks contact with the fingers.

The second and third clauses are dual of each other. Let's denote by $W^{<}_{f_x,f_y}$ the convexe of force directions which can be generated by W. Similarly, we denote $T^{<}_{v_x,v_y}$ the convexe of translational velocities of object B in the plane. We have seen in Section 2.4 that the convexes $W^{<}_{f_x,f_y}$ and $T^{<}_{v_x,v_y}$ are dual of each other. So $T^{<}_{v_x,v_y}$ *null* implies that $W^{<}_{f_x,f_y}$ is *total*, and vice versa. ∎

4.2 An Analytical View of Force-Direction Closure

The necessary and sufficient condition for a set of wrenches W to generate force with arbitrary direction is:

Theorem 4.2 *A set of wrenches W can generate force in any direction if and only if there exists a three-tuple of wrenches $\{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}\}$ whose respective force directions $\mathbf{f_1}, \mathbf{f_2}, \mathbf{f_3}$ satisfy:*

- *Two of the three directions $\mathbf{f_1}, \mathbf{f_2}, \mathbf{f_3}$ are independent.*

- *There exist α, β, γ all greater than zero, such that:*

$$\alpha \mathbf{f_1} + \beta \mathbf{f_2} + \gamma \mathbf{f_3} = \mathbf{0}$$

That is, a strictly positive combination of the three directions is zero.

Proof: We use the second clause of Theorem 4.1 to find the necessary and sufficient condition for which there is no translational twist reciprocal or repelling to W. No reciprocal or repelling translational twist means the system of linear inequations described by:

$$[\mathbf{W}]\ \mathbf{t^s} \geq \mathbf{0} \tag{12}$$

has no non-zero solution $\mathbf{t^s} = [v_x,\ v_y \mid 0]$. $\mathbf{t^s}$ is the spatial transpose of the twist $\mathbf{t}$.

Since a translational twist is a free vector with zero angular velocity, we get a reduced system of homogeneous linear inequations in only two unknowns v_x, v_y. For such system to have no solution, we must need at least three inequations, or W must have at least three wrenches (Hunt 1956, Strang 1976).

It is obvious that if no solution exists for some three-tuple of inequations of system (12), then no solution exists for system (12), and vice versa. Without loss of generality, let's assume that W contains exactly one such three-tuple $\{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}\}$. After dropping out the moment and angular velocity terms, system (12) reduces to:

$$\begin{pmatrix} f_{1x} & f_{1y} \\ f_{2x} & f_{2y} \\ f_{3x} & f_{3y} \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{13}$$

There is no homogeneous solution if and only if the 3x2 matrix $[W]$ is of rank 2, or if and only if two of the three force directions are non-parallel.

Assuming that there is no homogeneous solution, the rank of $[W]$ is $r = 2$. Any particular solution must be a 1-face ("r-1"-face, Hunt 1956) with a zero product with one row of $[W]$ and strictly positive products with the remaining rows of $[W]$. In other words, the necessary and sufficient condition for the existence of a particular solution is that the solution has a zero product with one row of $[W]$, and two non-zero products having the same sign with the two remaining rows of $[W]$. [3]

Conversely, there is no particular solution if and only if all 1-face vectors perpendicular to one row of $[W]$ have products of different signs with the remaining rows of $[W]$. Concretely, let's solve for the nonexistence of repelling translational velocity $\mathbf{v}$ reciprocal to the force direction $\mathbf{f_1}$:

$$\begin{pmatrix} f_{1x} & f_{1y} \\ f_{2x} & f_{2y} \\ f_{3x} & f_{3y} \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} 0 \\ \beta_1 \\ -\gamma_1 \end{pmatrix} \tag{14}$$

[3] In the case the two non-zero products are both negative, we can always negate the solution to make the non-zero products positive.

β_1, γ_1 are both of the same sign and non-zero.

From the first equation of system (14), we solve for v_x, v_y in terms of f_{1x}, f_{1y}, and replace them in the second and third equations to get an equation in terms of the three force directions $\mathbf{f_1}, \mathbf{f_2}, \mathbf{f_3}$. After simplifications, we get:

$$(\mathbf{f_2} \times \mathbf{f_3})\,\mathbf{f_1} + \beta_1\mathbf{f_2} + \gamma_1\mathbf{f_3} \;=\; 0 \tag{15}$$

By rotating the subscripts and coefficients, we get two other equations for the non-existence of repelling translational velocity which is respectively reciprocal to the force direction $\mathbf{f_2}$, and $\mathbf{f_3}$.

$$\alpha_2\mathbf{f_1} + (\mathbf{f_3} \times \mathbf{f_1})\,\mathbf{f_2} + \gamma_2\mathbf{f_3} \;=\; 0 \tag{16}$$

$$\alpha_3\mathbf{f_1} + \beta_3\mathbf{f_2} + (\mathbf{f_1} \times \mathbf{f_2})\,\mathbf{f_3} \;=\; 0 \tag{17}$$

In the above equations, (15) (16) (17), the coefficients α_i, β_i must have the same sign within each equation.

Without loss of generality, let's assume that the force directions $\mathbf{f_1}, \mathbf{f_2}, \mathbf{f_3}$ are ordered counter-clockwise, so that all the pairwise cross products are strictly greater than zero. Since we have assumed that two of the three force directions are independent, the third force direction can be uniquely expressed as a linear combination of the first two. This implies that the three equations (15), (16), and (17) all express one unique linear combination, describing the constraint that the positive combination of the three force directions is null. We conclude that: assuming two force directions are non parallel, there is no repelling translational velocity if and only if there exist α, β, γ all greater than zero, such that:

$$\alpha\mathbf{f_1} + \beta\mathbf{f_2} + \gamma\mathbf{f_3} \;=\; \mathbf{0} \tag{18}$$

∎

4.3 A Geometrical View of Force-Direction Closure

Theorem 4.2 can be captured in a more suggestive and compact way as follows:

Corollary 4.1 *A set of wrenches W can generate forces in any arbitrary direction if and only if there exists a three-tuple of force-direction vectors* $\mathbf{f_1}, \mathbf{f_2}, \mathbf{f_3}$ *whose end points draw a nonzero triangle that includes their common origin point.*

Proof: A bit of geometry will convince the reader that the above corollary is equivalent to Theorem 4.2. Alternatively, we can start from system (14) and solve for the product $\beta_1\gamma_1$ which is the product of two cross-products:

$$\beta_1\gamma_1 \;=\; (\mathbf{f_1} \times \mathbf{f_2})\,(\mathbf{f_3} \times \mathbf{f_1})$$

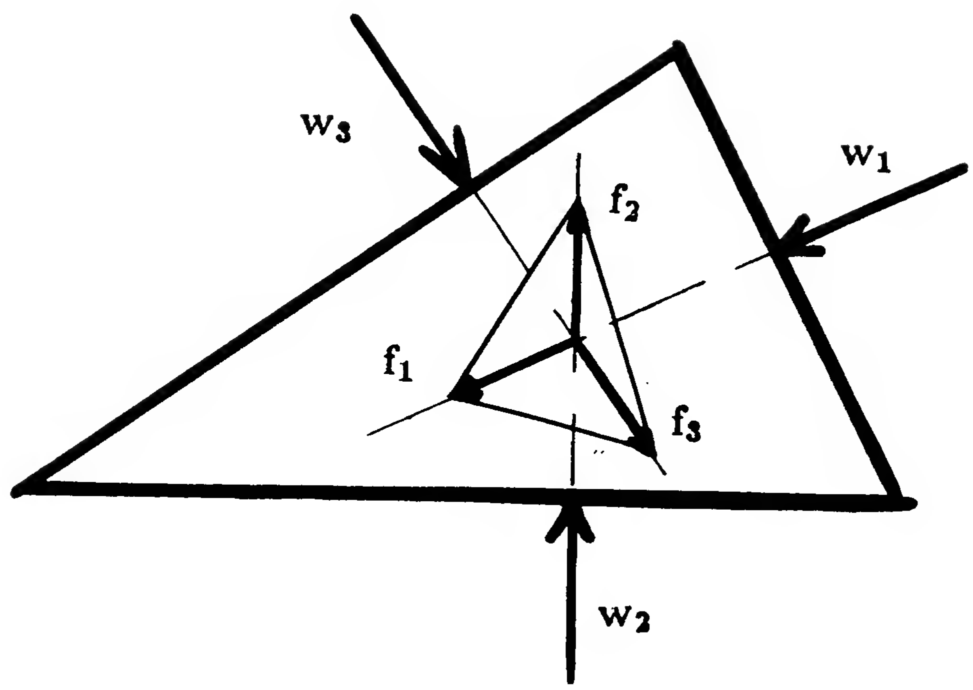


Figure 6: A geometrical view of force-direction closure.

Similarly, the other two products $\alpha_2\gamma_2$ and $\alpha_3\beta_3$ are:

$$\begin{aligned}\alpha_2\,\gamma_2 &= (\mathbf{f_1}\times\mathbf{f_2})\,(\mathbf{f_2}\times\mathbf{f_3})\\ \alpha_3\,\beta_3 &= (\mathbf{f_3}\times\mathbf{f_1})\,(\mathbf{f_2}\times\mathbf{f_3})\end{aligned}$$

Without loss of generality, we assume that the force directions are ordered counter-clockwise. Since the coefficients α_i, β_i all have the same sign and are non-zero, their pair-wise products and the cross-products must be strictly greater than zero. Recognizing the cross-product between two unit force directions as the sine of the angle between these two force directions, we can conclude that the three angles in between the three force directions must be strictly greater than 0 and less than π. This is nothing more than the picture of three vectors pointing outward from a common origin with their ending arrows drawing a triangle which includes this origin, see Figure 6. ∎

5 Ability To Resist Arbitrary Rotation

5.1 Torque Closure with Planar Forces

We now investigate the necessary and sufficient condition for a grasp to resist clockwise and counter-clockwise rotations of the object. First, let's look at three equivalent views of the same problem:

Theorem 5.1 *Let a grasp configuration G be described by the set of wrenches W. Each wrench in W is a contact force which acts on the object B. The three following clauses are equivalent:*

1. *Any non null rotational twist $\mathbf{t} = [\omega_z \,|\, \omega_z r_y, \, -\omega_z r_x]$ of the object B is contrary to at least one wrench of W.*
2. *There is no non null rotational twist either reciprocal or repelling to all wrenches in W.*
3. *The positive combination of the wrenches in W can generate clockwise and counter-clockwise torques. We say that grasp G has torque closure.*

Proof: The second clause is the double negation of the first clause, so they are equivalent. They both express our intuitive notion that saying "there is no free rotation nor a rotation that breaks contact" is equivalent to saying that "any rotation will be resisted by a contact force", exerted by some finger of the hand.

The second and third clauses are dual of each other. Let's denote by $W_{m_z}^{<}$ the convexe of torques which can be generated by W. Similarly, we denote by $T_{\omega_z}^{<}$ the convexe of rotations in the plane. From Section 2.4, we know that the two convexes $W_{m_z}^{<}$ and $T_{\omega_z}^{<}$ are dual of each other. So $T_{\omega_z}^{<}$ null implies $W_{m_z}^{<}$ total, and vice versa. ∎

5.2 An Analytical View of Torque Closure

Torque closure can be achieved by creating enough friction on some axis of rotation of the object. The friction between the rotating object and its supporting axis will create a torque which resists any clockwise or counter-clockwise rotation of the object. Unfortunately, in most grasp configurations, we have only point contacts, and through a point contact, a finger can exert only a pure force on the object and not torque. The interesting problem is how to achieve torque closure with only pure forces. The following theorem states the analytical necessary and sufficient condition for a set of contact forces to generate clockwise and counter-clockwise torques.

Theorem 5.2 *A set of planar forces W can generate clockwise and counter-clockwise torques if and only if there exists a four-tuple of forces $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4\}$ such that:*

- *Three of the four forces have lines of action that do not intersect at a common point or at infinity.*
- *Let $\mathbf{f}_1, \ldots, \mathbf{f}_4$ be the force directions of $\mathbf{w}_1, \ldots, \mathbf{w}_4$. Let $\mathbf{p}_{12}$ (resp. $\mathbf{p}_{34}$) be the point where the lines of action of $\mathbf{w}_1$ and $\mathbf{w}_2$ (resp. $\mathbf{w}_3$, and $\mathbf{w}_4$) intersect. There exist $\alpha, \beta, \gamma, \delta$ all greater than zero, such that:*

$$\begin{aligned} \mathbf{p}_{34} - \mathbf{p}_{12} &= \pm(\alpha \mathbf{f}_1 + \beta \mathbf{f}_2) \\ &= \mp(\gamma \mathbf{f}_3 + \delta \mathbf{f}_4) \end{aligned}$$

Proof: [The proof is quite long and has the same flavor as the proof of theorem 4.2. On first reading, the reader is advised to skip this proof and return to it later.]

We use the second clause of Theorem 5.1 to find the necessary and sufficient condition for which there is no rotational twist reciprocal or repelling to W. This means that the system of linear inequations described by:

$$[\mathbf{W}]\ \mathbf{t}^{s}\ \geq\ \mathbf{0} \tag{19}$$

has neither homogeneous nor particular solution. $\mathbf{t}^{s} = [\omega_z r_y,\ -\omega_z r_x \mid \omega_z]^t$ is the spatial transpose of the twist $\mathbf{t}$.

We get a system of homogeneous linear inequations in three unknowns. For such a system to have no solution, we need at least four inequations, or four wrenches. If no solution exists for some four-tuple of inequations from system (19), then no solution exists for system (19), and vice versa. So, without loss of generality, we assume that W is exactly one such four-tuple of wrenches.

There is no homogeneous solution if and only if the 4x3 matrix $[W]$ is of rank 3, or if and only if there is a 3x3 block from $[W]$ that has non zero determinant. Assume that the first three rows form such block. The determinant is:

$$det\,(\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}) \;=\; \begin{vmatrix} f_{1x} & f_{1y} & \mathbf{r_1} \times \mathbf{f_1} \\ f_{2x} & f_{2y} & \mathbf{r_2} \times \mathbf{f_2} \\ f_{3x} & f_{3y} & \mathbf{r_3} \times \mathbf{f_3} \end{vmatrix} \tag{20}$$

By expanding the determinant along the third column, we get:

$$\begin{aligned} det\,(\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}) \;=\;& (\mathbf{r_1} \times \mathbf{f_1})(\mathbf{f_2} \times \mathbf{f_3}) \\ &+ (\mathbf{r_2} \times \mathbf{f_2})(\mathbf{f_3} \times \mathbf{f_1}) \\ &+ (\mathbf{r_3} \times \mathbf{f_3})(\mathbf{f_1} \times \mathbf{f_2}) \end{aligned} \tag{21}$$

From the above equation, if the three lines of force are parallel with each other, then the three cross products of the force directions are zero, and so is the determinant. Let's assume that they are not all three parallel, and that the lines of action of $\mathbf{w_1}, \mathbf{w_2}$ intersect at $\mathbf{p_{12}}$. We can choose $\mathbf{p_{12}}$ as the origin of our reference frame. With this choice of origin, the moment components of the wrenches $\mathbf{w_1}, \mathbf{w_2}$ become zero, and so the first two terms in right hand side of equation (21) drop out. The determinant reduces to:

$$det\,(\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}) \;=\; ((\mathbf{r_3} - \mathbf{p_{12}}) \times \mathbf{f_3})\,(\mathbf{f_1} \times \mathbf{f_2}) \tag{22}$$

The determinant can be zero if and only if the first cross-product in equation (22) is zero, or if and only if the line of force of $\mathbf{w_3}$ also goes through $\mathbf{p_{12}}$. We conclude that there is no free rotation if and only if both the followings do not hold:

- The three lines of force intersect at a common point. In this case, the object B can freely rotate about the z-axis going through this common point.

- The three lines of force are all parallel. This case corresponds to a free translation perpendicular to the direction of the three forces. We can think of this translation as a rotation with rotation point at infinity.

Assuming that the 4x3 matrix $[W]$ is of rank 3, there is no particular solution to system (19) if and only if any 2-face vector orthogonal to two rows of $[W]$ has products of different signs with the remaining rows of $[W]$. Let's solve for the non-existence of rotational twist, reciprocal to the first two wrenches $\mathbf{w_1}, \mathbf{w_2}$, and repelling to the two last wrenches $\mathbf{w_3}, \mathbf{w_4}$:

$$\begin{pmatrix} f_{1x} & f_{1y} & \mathbf{r_1} \times \mathbf{f_1} \\ f_{2x} & f_{2y} & \mathbf{r_2} \times \mathbf{f_2} \\ f_{3x} & f_{3y} & \mathbf{r_3} \times \mathbf{f_3} \\ f_{4x} & f_{4y} & \mathbf{r_4} \times \mathbf{f_4} \end{pmatrix} \begin{pmatrix} \omega_z r_y \\ -\omega_z r_x \\ \omega_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \gamma_0 \\ -\delta_0 \end{pmatrix} \tag{23}$$

γ_0, δ_0 are both of the same sign and non zero.

Without loss of generality, let's factor out ω_z. Let P_{12}, (resp. P_{34}) be the point where the lines of force of $\mathbf{w_1}, \mathbf{w_2}$ (resp. $\mathbf{w_3}, \mathbf{w_4}$) intersect. From the first two equations, we solve for the point of rotation $\mathbf{r}$:

$$\begin{aligned} \mathbf{r} &= \frac{1}{\mathbf{f_1} \times \mathbf{f_2}} \left[(\mathbf{r_2} \times \mathbf{f_2})\, \mathbf{f_1} - (\mathbf{r_1} \times \mathbf{f_1})\, \mathbf{f_2} \right] \\ &= \mathbf{p_{12}} \end{aligned} \tag{24}$$

The above equation makes sense: the point of free rotation is the point where the two lines of force intersect. Similarly, from the third and fourth equations of system (23), we solve for the instantaneous center of rotation $\mathbf{r}$:

$$\begin{aligned} \mathbf{r} &= \frac{1}{\mathbf{f_3} \times \mathbf{f_4}} \left[(\mathbf{r_4} \times \mathbf{f_4})\, \mathbf{f_3} - (\mathbf{r_3} \times \mathbf{f_3})\, \mathbf{f_4} \right] \\ &\quad + \frac{1}{\mathbf{f_3} \times \mathbf{f_4}} (\gamma_0 \mathbf{f_3} + \delta_0 \mathbf{f_4}) \\ &= \mathbf{p_{34}} + \frac{1}{\mathbf{f_3} \times \mathbf{f_4}} (\gamma_0 \mathbf{f_3} + \delta_0 \mathbf{f_4}) \end{aligned} \tag{25}$$

Eliminating $\mathbf{r}$ from the two equations (24) (25), we find a constraint equation with the following form:

$$\mathbf{p_{12}} - \mathbf{p_{34}} = (\gamma_1\, \mathbf{f_3} + \delta_1\, \mathbf{f_4}) \tag{26}$$

where γ_1, δ_1 have both the same sign and non zero.

By rotating the numbers $1, \ldots, 4$ and the coefficients $\alpha, \ldots, \delta$, we get the equation expressing the nonexistence of repelling rotational twist $\mathbf{t^s}$ which is reciprocal to the wrenches $\mathbf{w_3}, \mathbf{w_4}$:

$$\mathbf{p_{34}} - \mathbf{p_{12}} = (\alpha_1\, \mathbf{f_1} + \beta_1\, \mathbf{f_2}) \tag{27}$$

We also get four other equations for the other two pairings $[(\mathbf{w_1}, \mathbf{w_3}), (\mathbf{w_2}, \mathbf{w_4})]$ and $[(\mathbf{w_1}, \mathbf{w_4}), (\mathbf{w_2}, \mathbf{w_3})]$:

$$\mathbf{p_{13}} - \mathbf{p_{24}} = (\beta_2 \mathbf{f_2} + \delta_2 \mathbf{f_4}) \tag{28}$$

$$\mathbf{p_{24}} - \mathbf{p_{13}} = (\alpha_2 \mathbf{f_1} + \gamma_2 \mathbf{f_3}) \tag{29}$$

$$\mathbf{p_{14}} - \mathbf{p_{23}} = (\beta_3 \mathbf{f_2} + \gamma_3 \mathbf{f_3}) \tag{30}$$

$$\mathbf{p_{23}} - \mathbf{p_{14}} = (\alpha_3 \mathbf{f_1} + \delta_3 \mathbf{f_4}) \tag{31}$$

We use the fact that the points P_{12}, P_{13}, P_{14} are on the same line of action of wrench $\mathbf{w_1}$, etc ... to prove that the above six equations (26)—(31) are satisfied if and only if all the coefficients $\alpha_i, \ldots, \delta_i$ are of the same sign. We are able to prove a stronger result which states that if one pair of equations like (26, 27) holds, with coefficients $\alpha_1, \ldots, \delta_1$ all of the same sign, then the other two pairs (28, 29) and (30, 31) hold, and vice versa. See Corollary 5.1.

With Corollary 5.1, we conclude that there is no rotational twist repelling to W if and only if any of the 3 pairs of equations (26, 27), (28, 29), (30, 31) hold. Namely, if and only if there exists a pairing such as $[(\mathbf{w_1}, \mathbf{w_2}), (\mathbf{w_3}, \mathbf{w_4})]$ with $\alpha, \beta, \gamma, \delta$ all greater than zero, such that:

$$\begin{aligned} \mathbf{p_{34}} - \mathbf{p_{12}} &= \pm(\alpha \mathbf{f_1} + \beta \mathbf{f_2}) \\ &= \mp(\gamma \mathbf{f_3} + \delta \mathbf{f_4}) \end{aligned} \tag{32}$$

Particular cases arise when the pairing $[(\mathbf{w_1}, \mathbf{w_2}), (\mathbf{w_3}, \mathbf{w_4})]$ has $\mathbf{w_1}$ parallel to $\mathbf{w_2}$, or $\mathbf{w_3}$ parallel to $\mathbf{w_4}$. We can avoid handling these particular cases by considering another pairing like $[(\mathbf{w_1}, \mathbf{w_3}), (\mathbf{w_2}, \mathbf{w_4})]$, or $[(\mathbf{w_1}, \mathbf{w_4}), (\mathbf{w_2}, \mathbf{w_3})]$. If we assume that the four forces in W span the space of all force directions, then we never get three forces that are parallel with each other. So there is always at least two pairings that work to prove the nonexistence of rotational twists repelling to W if the grasp has torque closure. ■

To complete the discussion of this section, we state and prove Corollary 5.1 which allows us to consider only one pairing instead of all three possible pairings:

Corollary 5.1 *Let four lines with directions* $\mathbf{f_1}, \mathbf{f_2}, \mathbf{f_3}, \mathbf{f_4}$ *intersect pairwise at six points* $P_{12}, \ldots, P_{34}$.

$$\begin{aligned} \mathbf{p_{34}} - \mathbf{p_{12}} &= (\alpha_1 \mathbf{f_1} + \beta_1 \mathbf{f_2}) \\ &= -(\gamma_1 \mathbf{f_3} + \delta_1 \mathbf{f_4}) \\ \mathbf{p_{24}} - \mathbf{p_{13}} &= (\alpha_2 \mathbf{f_1} + \gamma_2 \mathbf{f_3}) \\ &= -(\beta_2 \mathbf{f_2} + \delta_2 \mathbf{f_4}) \\ \mathbf{p_{23}} - \mathbf{p_{14}} &= (\alpha_3 \mathbf{f_1} + \delta_3 \mathbf{f_4}) \\ &= -(\beta_3 \mathbf{f_2} + \gamma_3 \mathbf{f_3}) \end{aligned} \tag{33}$$

The above 6 equations all have Greek coefficients with the same sign within each equation (not necessarily across all six equations) if and only if $\alpha_i, \beta_i, \gamma_i, \delta_i$ all have the same sign for either $i = 1$, or 2, or 3.

Proof: Let's assume that we have the first two equations:

$$\begin{aligned} \mathbf{p_{34}} - \mathbf{p_{12}} &= (\alpha_1 \mathbf{f_1} + \beta_1 \mathbf{f_2}) \\ &= -(\gamma_1 \mathbf{f_3} + \delta_1 \mathbf{f_4}) \end{aligned} \tag{34}$$

with $\alpha_1 > 0$, $\beta_1 > 0$, and $\gamma_1 \delta_1 > 0$. We'll prove that the four coefficients $\alpha_1, \beta_1, \gamma_1, \delta_1$ are all greater than zero, that is, we have the scenario illustrated in Figure 7. We compute the intersection points $\mathbf{p_{23}}$ and $\mathbf{p_{14}}$:

$$\begin{aligned} \mathbf{p_{23}} &= \mathbf{p_{12}} + \frac{\mathbf{p'} \times \mathbf{f_3}}{\mathbf{f_2} \times \mathbf{f_3}} \mathbf{f_2} \\ &= \mathbf{p_{34}} + \frac{\mathbf{p'} \times \mathbf{f_2}}{\mathbf{f_2} \times \mathbf{f_3}} \mathbf{f_3} \\ \mathbf{p_{14}} &= \mathbf{p_{12}} + \frac{\mathbf{p'} \times \mathbf{f_4}}{\mathbf{f_1} \times \mathbf{f_4}} \mathbf{f_1} \\ &= \mathbf{p_{34}} + \frac{\mathbf{p'} \times \mathbf{f_1}}{\mathbf{f_1} \times \mathbf{f_4}} \mathbf{f_4} \end{aligned} \tag{35}$$

where $\mathbf{p'} = \mathbf{p_{34}} - \mathbf{p_{12}}$. Next, we compute the expression for $\mathbf{p_{23}} - \mathbf{p_{14}}$:

$$\begin{aligned} \mathbf{p_{23}} - \mathbf{p_{14}} &= -\gamma_1 \frac{\mathbf{f_3} \times \mathbf{f_4}}{\mathbf{f_4} \times \mathbf{f_1}} \mathbf{f_1} + \delta_1 \frac{\mathbf{f_3} \times \mathbf{f_4}}{\mathbf{f_2} \times \mathbf{f_3}} \mathbf{f_2} \\ &= \alpha_1 \frac{\mathbf{f_1} \times \mathbf{f_2}}{\mathbf{f_2} \times \mathbf{f_3}} \mathbf{f_3} - \beta_1 \frac{\mathbf{f_1} \times \mathbf{f_2}}{\mathbf{f_4} \times \mathbf{f_1}} \mathbf{f_4} \end{aligned} \tag{36}$$

Expressing $\mathbf{p_{23}} - \mathbf{p_{14}}$ in terms of linear combination of $\{\mathbf{f_2}, \mathbf{f_3}\}$ is difficult. Instead of proving that there exist β_3, γ_3 non zero and of the same sign such that:

$$\mathbf{p_{23}} - \mathbf{p_{14}} = \beta_3 \mathbf{f_2} + \gamma_3 \mathbf{f_3}$$

we prove the equivalent: the vector $\mathbf{p_{23}} - \mathbf{p_{14}}$ has opposite sign cross-products with the vectors $\mathbf{f_2}, \mathbf{f_3}$, i.e:

$$[(\mathbf{p_{23}} - \mathbf{p_{14}}) \times \mathbf{f_2}][(\mathbf{p_{23}} - \mathbf{p_{14}}) \times \mathbf{f_3}] < 0$$

From equations (36), we get:

$$[(\mathbf{p_{23}} - \mathbf{p_{14}}) \times \mathbf{f_2}][(\mathbf{p_{23}} - \mathbf{p_{14}}) \times \mathbf{f_3}] = -\beta_1 \gamma_1 \frac{(\mathbf{f_1} \times \mathbf{f_2})^2 (\mathbf{f_3} \times \mathbf{f_4})^2}{(\mathbf{f_4} \times \mathbf{f_1})^2} \tag{37}$$

We deduce that the necessary and sufficient condition for the two last equations of (33) to hold is that β_1 be of the same sign with γ_1. We extrapolate this partial proof and argue that:

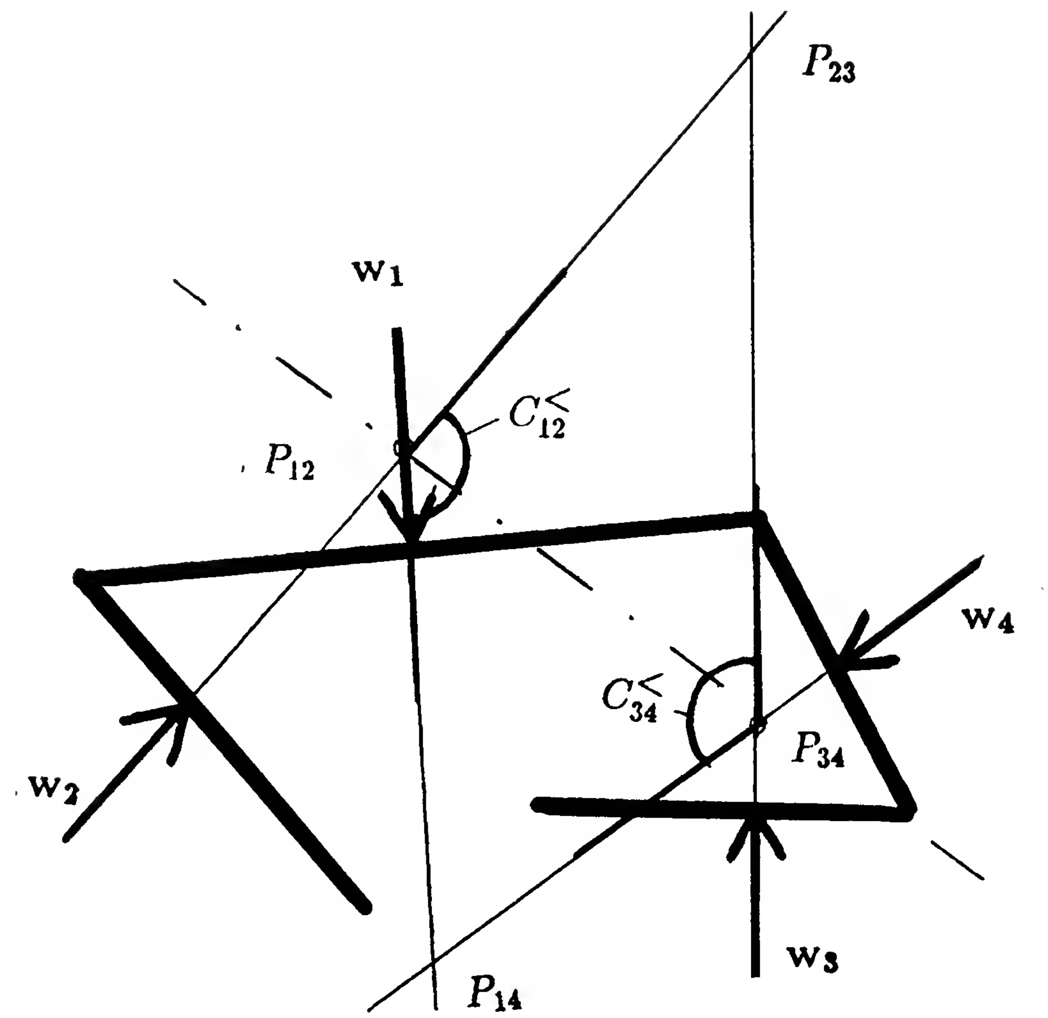


Figure 7: A geometrical view of torque closure.

- [$\Rightarrow$] The fact that the six equations of (33) hold implies that $\alpha_i, \ldots, \delta_i$ all have the same sign for $i = 1$, or 2, or 3. We have proved this implication for $i = 1$ using Equation (37). Similar proofs exist for i = 2 and 3.

- [$\Leftarrow$] From Equation (37), if $\alpha_1, \ldots, \delta_1$ all have the same sign then:

$$\mathbf{p_{23}} - \mathbf{p_{14}} = \beta_3 \mathbf{f_2} + \gamma_3 \mathbf{f_3}$$

with $\beta_3\gamma_3 > 0$. Equations similar to (37) allow us to deduce that all the six equations in (33) must hold.

∎

5.3 A Geometrical View of Torque Closure

It is useful to formulate Theorem 5.2 in more geometrical terms. The following corollary captures both constraints of Theorem 5.2 (Figure 7):

Corollary 5.2 *A set of planar forces W can generate clockwise and counter-clockwise torques if and only if there exists a four-tuple of forces $\{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}, \mathbf{w_4}\}$ such that the segment $P_{12}P_{34}$, points out of and into the 2 cones $C_{12}^{<}$, $C_{34}^{<}$, formed by the two pairs $(\mathbf{w_1}, \mathbf{w_2})$, and $(\mathbf{w_3}, \mathbf{w_4})$.*

Proof:

- The fact that the 4 forces form two non-null cones $C_{12}^{<}$, $C_{34}^{<}$ guarantees that no three forces are parallel. Next, the fact that the segment $P_{12}P_{34}$ is non zero guarantees that the four forces do not intersect at a common point. This is the first constraint of Theorem 5.2.

- Let's look at:

$$\begin{aligned} \mathbf{p_{34}} - \mathbf{p_{12}} &= \alpha\,\mathbf{f_1} + \beta\,\mathbf{f_2} \\ &= -(\gamma\,\mathbf{f_3} + \delta\,\mathbf{f_4}) \end{aligned}$$

  where all the Greek coefficients are strictly positive. Note that $\alpha\mathbf{f_1} + \beta\mathbf{f_2}$ is the cone of rays bounded by the force directions $\mathbf{f_1}, \mathbf{f_2}$, that is $C_{12}^{<}$. Similarly, $C_{34}^{<}$ is described by $\gamma\mathbf{f_3} + \delta\mathbf{f_4}$. The segment $P_{12}P_{34}$ points 'out of' $C_{12}^{<}$ (resp. 'into' $C_{34}^{<}$) if and only if $\mathbf{p_{34}} - \mathbf{p_{12}}$ it is a positive (resp. negative) combination of the force directions $(\mathbf{f_1}, \mathbf{f_2})$ (resp. $(\mathbf{f_3}, \mathbf{f_4})$). This is the second constraint of Theorem 5.2.

∎

From Figure 7, the reader can check for torque closure in the plane by drawing a parallelogram inside the overlapping region of the two cones $C_{12}^{<}$, $C_{34}^{<}$. From this parallelogram, he can generate clockwise and counter-clockwise torques from non-negative combination of the four pure forces.

6 Finding Force Closure Grasps

We have seen from Section 2.4 that the space of planar twists and wrenches are dual of each other. The following theorem formally states the force closure constraint in the plane:

Theorem 6.1 *Let G be a planar grasp described by the set of wrenches W. Let's denote by $W^{<}$ the wrench convexe spanned by W, and by $T^{<}$ the twist convexe reciprocal or repelling to W. The following clauses are equivalent:*

1. *G is a force closure grasp.*
2. *W can generate force with arbitrary direction, and moment. Formally:*

$$W^{<} = \infty\,[f_x,\, f_y \mid m_z]$$

3. *There is neither translational nor rotational twist that is free, or that breaks contact with G. Formally:*

$$T^{<} = \mathbf{0}\,[\omega_z \mid v_x,\, v_y]$$

We know from Section 2.4 that the convex addition of the convexe of all force directions $\infty|f_x, f_y|$ and the convexe of all torques $\infty|m_z|$ is the convexe of all planar forces $\infty|f_x, f_y \mid m_z|$. So, from the above theorem, the necessary and sufficient condition for force closure is contained in both Theorems 4.2 and 5.2. If we assume that through any contact we can only exert force and not torque, then Theorem 5.2 subsumes Theorem 4.2. Thus Corollary 5.2 also describes the geometrical necessary and sufficient condition for force closure with planar forces only.

6.1 Frictionless Grasps on Four Edges

Without friction, a finger can exert only a pure force, going through the point of contact and perpendicular to the edge (Figure 8). In this case, to have force closure, we need at least four point contacts. Let's start by giving the algorithm for constructing a force closure grasp with four contacts on four edges of an object B.

Algorithm 6.1 *A force closure grasp between four edges $e_1, \ldots, e_4$ can be constructed as follows:*

1. *Pair up two edges e_1, e_2 against e_3, e_4 such that the two sectors $\mathcal{C}_{12}$, $\mathcal{C}_{34}$ are non null. By sector $\mathcal{C}_{12}$, we denote the smallest sector between the normals $-\mathbf{n_1}$, $-\mathbf{n_2}$. Similarly for sector $\mathcal{C}_{34}$.*
2. *Check that the two sectors $\mathcal{C}_{12}$, $\mathcal{C}_{34}$ counter-overlap, i.e:*
$$\mathcal{C}_{12} \bigcap -\mathcal{C}_{34} \neq \emptyset \tag{38}$$
3. *Find the parallelogram Π_{12} by intersecting the two infinite bands perpendicular to and containing the edges e_1 and e_2. Parallelogram Π_{12} is the locus of points P_{12} where the lines of force of $\mathbf{w_1}, \mathbf{w_2}$ intersect. Similarly, we find the parallelogram Π_{34} which represents the locus of points P_{34} where lines of force of $\mathbf{w_3}, \mathbf{w_4}$ intersect.*
4. *Pick two points P_{12}, P_{34} respectively from the parallelograms Π_{12}, Π_{34}, such that the direction of the line joining P_{12} and P_{34} is in the counter-overlapping sector $\mathcal{C} = \mathcal{C}_{12} \bigcap -\mathcal{C}_{34}$.*
5. *From point P_{12}, backproject along the normal $\mathbf{n_1}$, (resp. $\mathbf{n_2}$), to find the grasp point P_1, (resp. P_2), on edge e_1, (resp. e_2). Similarly, we find the grasp points P_3, and P_4 by backprojecting P_{34} respectively along the normals $\mathbf{n_3}, \mathbf{n_4}$.*
6. *The four grasp points P_1, P_2, P_3, P_4 found as above form a force closure grasp $G(P_1, P_2, P_3, P_4)$ between the four edges.*

Note that for each point P_{12}, we can find a convex region of points P_{34} by intersecting the parallelogram Π_{34} with the two-sided cone $C^{\times}(P_{12}, \mathcal{C})$ having vertex P_{12}, and sector $\mathcal{C}$. The two-sided cone is the combination of two fields of view from point P_{12} with sectors $\mathcal{C}$ and $-\mathcal{C}$. The field of view is defined as follows:

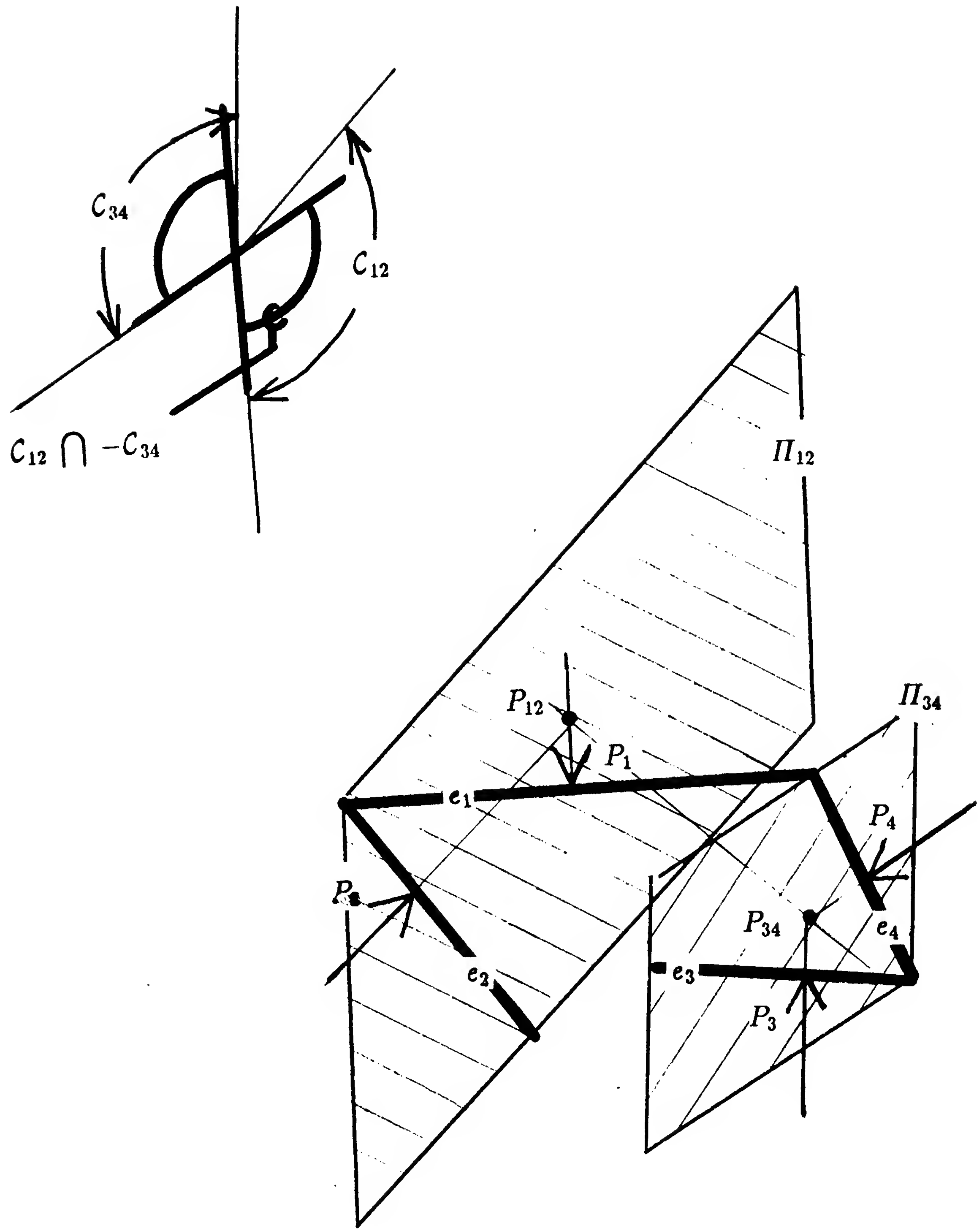


Figure 8: Finding frictionless grasps on four edges.

Definition 6.1 *The field of view of a figure A with sector $\mathcal{C}$ is the minimal cone $C^{\angle}(I, \mathcal{C})$ such that any ray from vertex I with direction in sector $\mathcal{C}$ will cut the figure A. The field of view is open-ended and one-sided.*

We can cut the parallelogram Π_{34} with the field of view of Π_{12} and sectors $\pm\mathcal{C}$. We get a convex region $\mathcal{R}_{34}$ which represents the set of points P_{34} which gives a force closure grasp with at least a point in Π_{12}. For example, the field of view of Π_{12} and sector $\mathcal{C}$ includes the parallelogram Π_{34}, and so $\mathcal{R}_{34}^{+} = \Pi_{34}$. Using sector $-\mathcal{C}$ will result in a smaller convex region $\mathcal{R}_{34}^{-}$, and $\mathcal{R}_{34} = \mathcal{R}_{34}^{+} \cup \mathcal{R}_{34}^{-}$.

Similarly, we find the convex region $\mathcal{R}_{12}$ which represents the set of points P_{12} which gives a force closure grasp with at least a point in Π_{34}. This time the respective sectors of view is $\mp\mathcal{C}$. From the construction, any point P_{12} in $\mathcal{R}_{12}$ will have at least one corresponding point P_{34} with which we have a force closure grasp. It is clear that:

Corollary 6.1 *There exists no force closure grasp between four edges $e_1, \ldots, e_4$ if and only if either of the following holds:*

1. *The two non null sectors $\mathcal{C}_{12}$, and $\mathcal{C}_{34}$ do not counter-overlap, i.e.:*

$$\mathcal{C} = \mathcal{C}_{12} \bigcap -\mathcal{C}_{34} = \emptyset$$

2. *The two fields of view of the parallelograms Π_{12} and Π_{34}, with respective sectors $\pm\mathcal{C}$ and $\mp\mathcal{C}$, are completely offset one from the other, i.e.:*

$$\begin{aligned} \mathcal{R}_{12} &= \emptyset \\ \textit{or}\ \mathcal{R}_{34} &= \emptyset \end{aligned}$$

The first constraint is called the *force-direction* constraint, expressing the condition for force-direction closure. The second constraint describes the condition for torque closure. We call the second constraint the *field-of-view* constraint, because of the way the regions $\mathcal{R}_{12}$, $\mathcal{R}_{34}$ are constructed. The force-direction constraint is more constraining than the field-of-view constraint when the object has perpendicular edges like rectangular blobs, and is less constraining otherwise. For example, a grasp on a rectangle needs all four different normals for force-direction closure, where as a grasp on a triangle needs only three normals.

The following theorem gives a geometrical representation of the set of force closure grasps, that is, the grasp-set from 4 edges.

Theorem 6.2 *The set of all possible grasps on four edges $e_1, \ldots, e_4$, denoted $\mathcal{G}(e_1, \ldots, e_4)$, is completely described by the two parallelograms Π_{12}, Π_{34}, and the counter-overlapping sector $\mathcal{C} = \mathcal{C}_{12} \bigcap -\mathcal{C}_{34}$, defined as above.*

$$\mathcal{G}(e_1, \ldots, e_4) = \langle \Pi_{12}, \Pi_{34}, \mathcal{C}_{12} \bigcap -\mathcal{C}_{34} \rangle \tag{39}$$

We can restrict the parallelograms Π_{12}, Π_{34}, by applying the field-of-view constraint, and describe the grasp set with the two convex regions $\mathcal{R}_{12}$, $\mathcal{R}_{34}$, constructed as above:

$$\mathcal{G}(e_1, \ldots, e_4) = \langle \mathcal{R}_{12}, \mathcal{R}_{34}, \mathcal{C}_{12} \bigcap -\mathcal{C}_{34} \rangle \tag{40}$$

Proof: It is obvious from the construction, and from Corollary 5.2 that the set of grasps characterized as above is complete for the pairing of edges e_1, e_2 against e_3, e_4. The reader may wonder whether the different pairings in step 1 of the above construction gives different sets of grasps. The answer is no.

Different pairings certainly give different descriptions for the grasp-set, because we get different parallelograms and counter-overlapping sectors. However, they all describe the same grasp-set. This is supported by Corollary 5.1, which says informally that the three pairings are equivalent to each other. So, the above description is complete. ∎

Complexity 6.1 *Let B be the object grasped with four frictionless point contacts:*

- *Finding the grasp-set or a force closure grasp between four edges costs constant time*
- *There are $O|edges(B)|^4$ 4-tuples of edges of object B. So, Enumerating all the force closure grasp-sets of object B costs $O|edges(B)|^4$.*

6.2 Frictionless Grasps on Three Edges

We have seen from Corollary 4.1 that to have force-direction closure we must have at least three non-parallel forces. So we need at least four contacts on three non-parallel edges, if there is no friction between the fingers and the grasped object. With two of the four contacts on the same edge, there are possibly three grasp-sets between three edges e_1, e_2, e_3. (Figure 9).

$$\mathcal{G}(e_1, e_2, e_1, e_3)$$
$$\mathcal{G}(e_1, e_2, e_2, e_3)$$
$$\mathcal{G}(e_1, e_3, e_2, e_3)$$

The problem formulated as above reduces to the problem of finding grasp-sets between four edges.

From Section 3.1, we can replace the two frictionless point contacts on the common edge with a frictionless edge contact. This is a good illustration of how we can grasp a same object with fewer fingers by using edge contacts instead of point contacts. We'll see how friction and soft contacts help even more in the next subsection.

Figure 10 shows a stable grasp with three spring contacts on three edges. The fingers act as springs pressing exactly at the places where the inscribed circle of the three edges is tangent to the three edges (Baker et al. 1985). This grasp configuration corresponds to a local minimum of the potential function of the three springs, so the grasp is stable. However, the object can still instantaneously rotate about the center of the inscribed circle with no opposing torque from the grasp. Note that we cannot generate a torque about this center of rotation, from the contact forces at the springs. The grasp is not force closure although it is stable for arbitrary small motions of the grasped object.

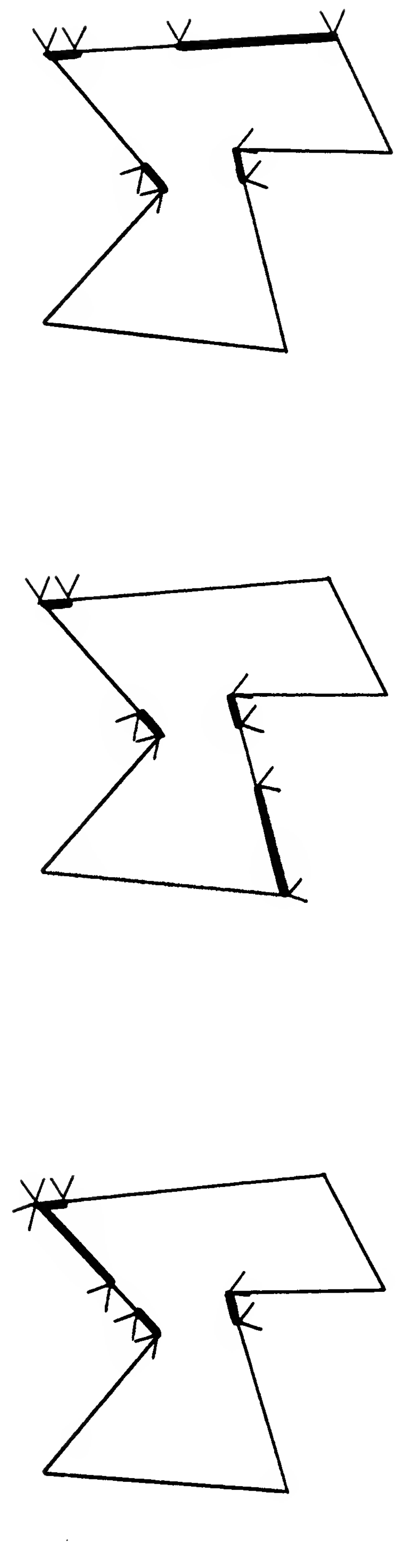


Figure 9: Frictionless grasps on three edges.

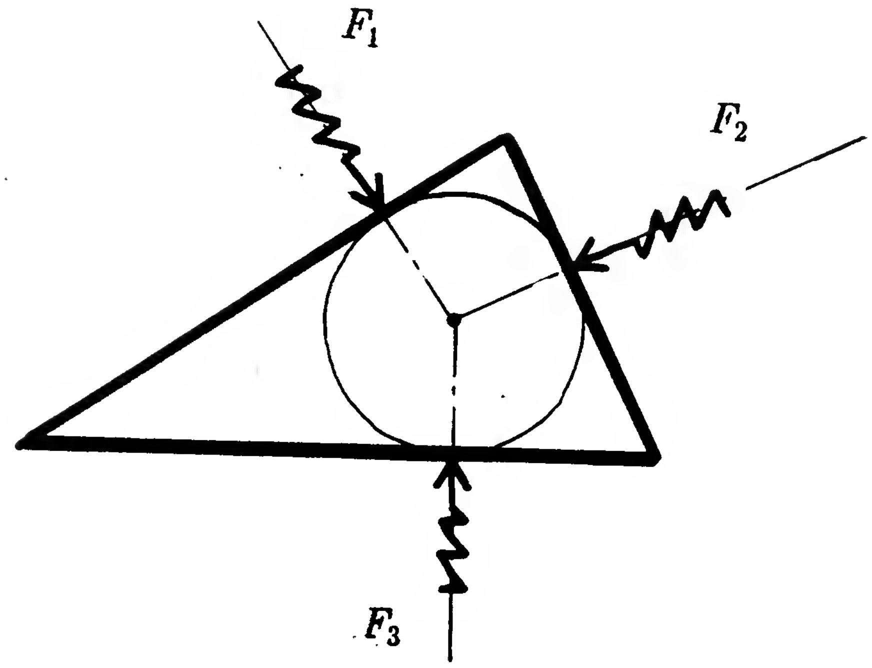


Figure 10: A stable grasp with three spring contacts.

6.3 Grasps on Two Edges Require Friction

Force closure grasps with only two point contacts instead of four require friction between the finger and the object. We have seen from subsection 3.1 that the existence of friction means the ability to resist against any force pointing into the friction cone. In other words, a single pushing force pointing into the friction cone can resist a range of forces described by the friction cone at the point of contact (Figure 4.b).

Mathematically, this friction cone can be viewed as the positive combination of the two extreme rays of the cone. So, a point contact with friction can be seen as equivalent to two contact forces. Two point contacts with friction is equivalent to four contact forces and so can result in a force closure grasp. The following theorem states the necessary and sufficient condition for force closure from two point contacts with friction:

Theorem 6.3 *Two point contacts with friction at P and Q is a force closure grasp if and only if the segment PQ points out of and into the two friction cones respectively at P and Q.*

Proof: This is a well known fact of planar mechanics. Let's however prove the above theorem using a reduction from a grasp with 2 point contacts with friction to a grasp with 4 point contacts without friction.

The reader will recognize that a friction cone at P, (resp. Q), is equivalent to two forces $\mathbf{w_1}, \mathbf{w_2}$, (resp. $\mathbf{w_3}, \mathbf{w_4}$), along the edge of friction cone and going through P, (resp. Q). We then recognize that point P, (resp. Q), is nothing more than the

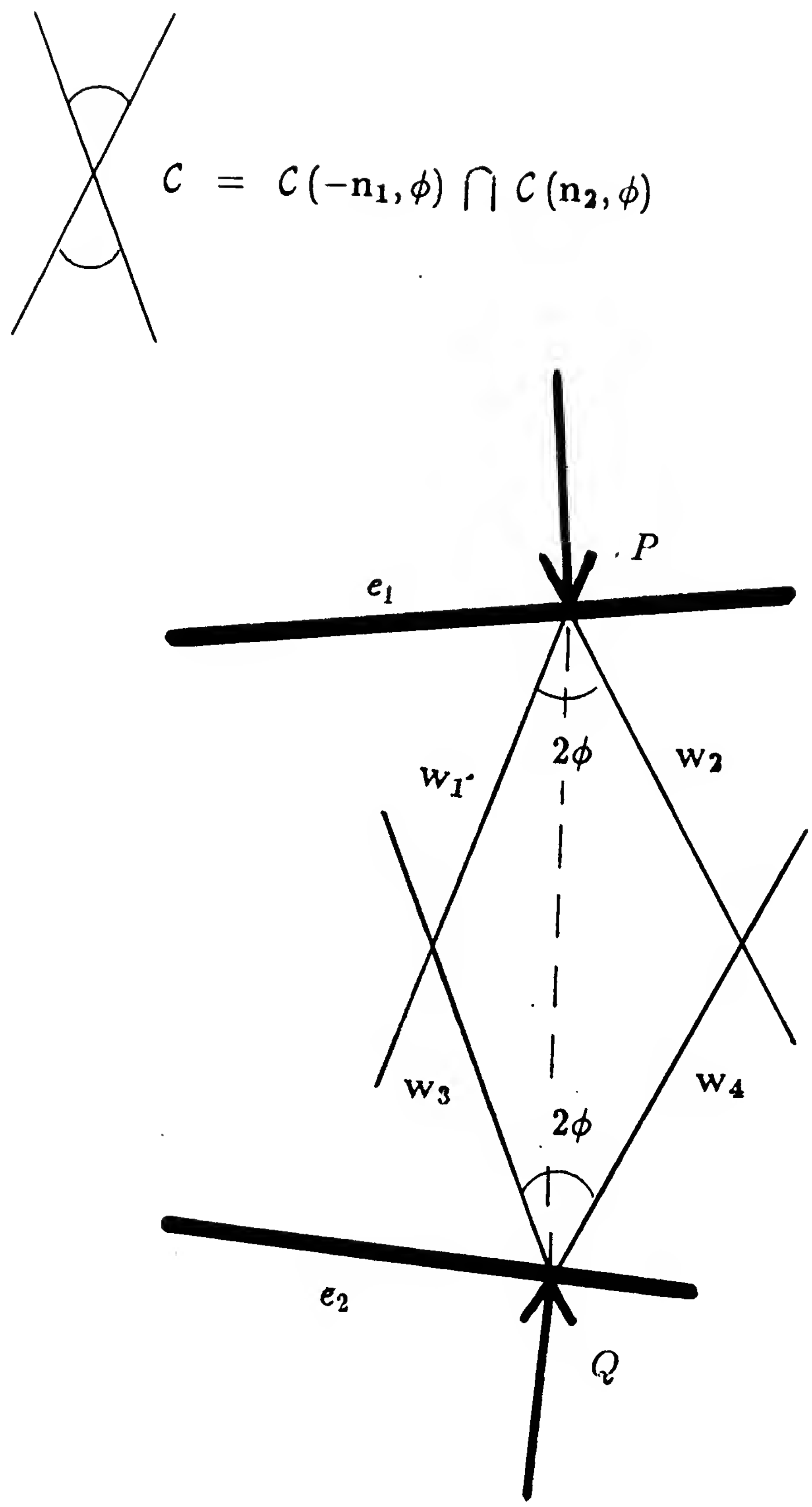


Figure 11: Finding grasps with friction on two edges.

point P_{12}, (resp. P_{34}). So the above theorem is the reformulation of Corollary 5.2. ∎

Now, let's find the set of possible grasps from two edges e_1, and e_2. Since the point of contact P, (resp. Q), must lie on edge e_1, (resp. e_2), the parallelogram II_{12}, (resp. II_{34}) reduces to the edge e_1, (resp. e_2). We can restrict the edge e_1, (resp. e_2), to the segment e'_1, (resp. e'_2), by intersecting it with the field of view of e_2, (resp. e_1) with sector $\pm\mathcal{C}$ (resp. $\mp\mathcal{C}$). For the example in Figure 11, we have $e'_1 = e_1$ and $e'_2 = e_2$. The construction of the grasp-set from two edges e_1, e_2 is similar to the construction given in Algorithm 6.1.

Theorem 6.4 *The set of all possible grasps with friction ϕ on two edges e_1, e_2, denoted $\mathcal{G}(e_1, e_2)$, is completely described by the two edges e_1, e_2, and the counter-overlapping sector:*

$$\mathcal{C} \;=\; \mathcal{C}(-\mathbf{n_1}, \phi) \bigcap \mathcal{C}(\mathbf{n_2}, \phi)$$

of the two friction cones resp. from edge e_1 and e_2.

$$\mathcal{G}(e_1, e_2) \;=\; \langle e_1,\, e_2,\, \mathcal{C}\rangle \tag{41}$$

We can restrict the edges e_1, e_2, by applying the field-of-view constraint, and describe the grasp-set by the two segments e'_1, e'_2, as follows:

$$\mathcal{G}(e_1, e_2) \;=\; \langle e'_1,\, e'_2,\, \mathcal{C}\rangle \tag{42}$$

How does soft finger contact compare with point contact with friction? Due to the larger area of contact, we see that a soft finger contact gives us a larger range of forces and moments showed by a range of friction cones instead of a single friction cone (Figure 4.e). A more interesting comparison is to compare the range of force directions. A soft finger contact at a vertex has a much larger range of force directions. The soft finger contact can be approximated as a point contact with a much wider friction cone. From Theorem 6.3, we have seen that the larger are the friction cones at the points of contacts, the greater is the likelihood that they 'see' each other, that is the grasp is force closure. So a soft finger gives us even more flexibility than a point contact with friction. This partially explains why people grasp objects at edges and corners, and also why the contacting surface of human fingers had better be soft rather than hard as nails.

7 Finding Independent Regions of Contact

7.1 Optimality Criterion

The previous section shows how to compute the set of all possible force closure grasps on a set of edges. In this set, we can look for an optimal grasp. The optimal grasp can be the grasp that requires the least amount of work from the fingers to

counteract the effect of gravity, or the effect of the moment of inertia, or some a priori assumed disturbance, etc...

In task planning, an alternative definition of optimality is to best deal with uncertainty and errors. We are interested in finding grasps that require as little accuracy as possible. One aspect of that goal is to have grasps such that the fingers can be positioned independently from each other not at discrete points, but within large regions of the edges. Formally:

Definition 7.1 *Let $\mathcal{G}$ be a grasp set on n edges. Finger F_i contacts at point P_i in region r_i of edge e_i, $i = 1, \ldots, n$. $\mathcal{G}$ is a grasp set with n independent regions of contact $r_1, \ldots, r_n$ if and only if:*

$$\forall P_1 \in r_1, \ldots, \forall P_n \in r_n, \quad G(P_1, \ldots, P_n) \textit{ is a force closure grasp}$$

$\mathcal{G}(r_1, \ldots, r_n)$ is a grasp-set with optimal independent regions of contact for the set of edges $\{e_1, \ldots, e_n\}$ if and only if:

$$\begin{aligned}
&\min(\|r_1\|, \ldots, \|r_{n-1}\|, \|r_n\|) \ \textit{is maximal},\\
&\min(\|r_1\|, \ldots, \|r_{n-1}\|) \ \textit{is maximal}, \ldots\\
&\min(\|r_1\|, \|r_2\|) \ \textit{is maximal}.
\end{aligned}$$

where $r_i \subseteq e_i$. The contact segments are ordered by decreasing length.

The next subsection uses Theorem 6.3 to cast the problem of finding the optimal grasp sets on two edges into a problem of fitting a two-sided cone cutting these two edges into two segments of largest minimum length. Similarly, using Corollary 5.2, the problem of finding the optimal grasp set on three or four edges becomes a problem of fitting a two-sided cone between 2 parallelograms.

7.2 Optimal Grasps With Friction

We give the algorithm for fitting a two-sided cone $C^\times(I, \mathcal{C})$ such that $C^\times(I, \mathcal{C})$ cuts the edges e_1 and e_2 into two segments e_1' and e_2' with largest minimum length:

Algorithm 7.1 *The optimal set of grasps on two edges e_1 and e_2 can be constructed as follows:*

1. *Find the two-sided cone $C^\times(I_1, \mathcal{C})$ that cuts all of edge e_1 and very little or none of edge e_2. We get a triangle $\triangle_1$ formed by edge e_1 and vertex I_1. This triangle represents the set of vertices I, where the two-sided cone $C^\times(I, \mathcal{C})$ monotonically cuts larger segment e_1' and smaller segment e_2' as we move from edge e_1 to e_2. Similarly, we find the two-sided cone $C^\times(I_2, \mathcal{C})$ such that this later cuts exactly the edge e_2 and very little or none of edge e_1. We get a triangle $\triangle_2$ formed by edge e_2 and vertex I_2.*
2. *Find the trade-off region for vertex I by intersecting the triangle $\triangle_1$ with $\triangle_2$.*

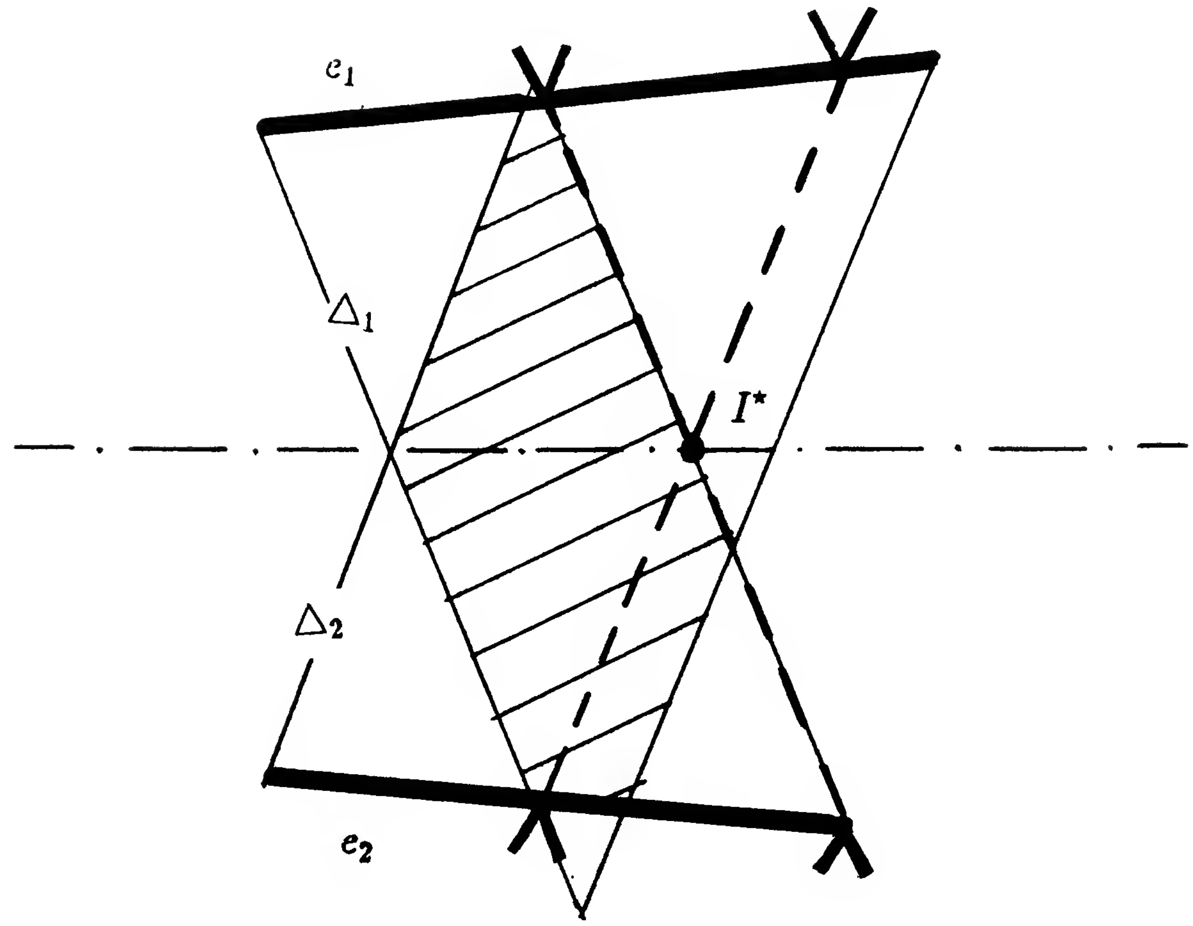


Figure 12: Finding the optimal set of grasps on two edges.

3. *We cut the trade-off region with the bisector of the two edges* e_1, *and* e_2. *The optimal vertex is at one of the two endpoints of the intersecting segment, or anywhere on this segment, depending on the direction of the cone formed by the two edges. If no intersection exists, then the optimal vertex is the point of the trade-off region which is nearest to the bisector.*

Proof: Figure 12 illustrates the different steps of the algorithm. Depending on the way the upper cone cuts edge e_1, we can partition the plane into 5 regions:

1. The cone cuts in the interior of edge e_1. This region is triangle Δ_1 with one of its sides being e_1. The length of the segment cut by the cone varies linearly with the distance of the vertex I_1 to edge e_1. In other words, the loci of I_1 whose cone cuts e_1 with constant length form segments parallel to e_1.

2. Only the left ray of the cone cuts edge e_1. The loci of points I_1 whose cone cuts e_1 with constant length form a set of line parallel to the left ray of the cone and are continuation on the right of the loci found in region 1.

3. Only the right ray of the cone cuts edge e_1. The loci are now the continuation on the left of the loci found in region 1.

4. The cone includes the whole edge e_1 in its field of view. This region corresponds to the ability of putting a finger anywhere on the whole edge e_1.

5. The one-sided cone does not cut edge e_1.

The above algorithm handles the general case where the two triangles $\triangle_1$ and $\triangle_2$ intersect. The trade-off region is the intersection $\triangle_1 \bigcap \triangle_2$. Cutting this trade-off region with the bisector gives us the locus of vertex I whose two-sided cone cuts equal segments on e_1 and e_2. The optimal vertex of the cone can be either anywhere on this locus or at the two end points of this locus, depending respectively on whether e_1 is parallel to e_2 or not. The case where the two triangles do not intersect can be handled as easily. ∎

Complexity 7.1 *Let B be the object grasped with two point contacts with friction:*

- *Finding the optimal set of grasps with independent regions of contact on two edges costs constant time.*
- *Enumerating the optimal set of grasps for all pairs of edges costs $O|edges(B)|^2$. So, finding the optimal pair of edges and its corresponding set of grasps costs $O|edges(B)|^2$. The pair of contact regions found has the largest minimum length.*

7.3 Optimal Grasps Without Friction

Theorem 7.1 *$\mathcal{G}(r_1, r_2, r_3, r_4)$ is a grasp-set with independent regions of contact $r_1, \ldots, r_4$ on edges $e_1, \ldots, e_4$ if and only if there exists a two-sided cone*

$$C^{\times}\left(I,\ \mathcal{C}_{12} \bigcap -\mathcal{C}_{34}\right)$$

which splits the two parallelograms Π'_{12}, Π'_{34} apart. Π'_{12}, (resp. Π'_{34}) is the restricted parallelogram generated by the regions r_1, r_2, (resp. r_3, r_4), of edges e_1, e_2, (resp. e_3, e_4).

Figure 13 illustrates the scenario described by the above theorem. The reader can easily prove the above theorem from Corollary 5.2 and the definition of a grasp with independent regions of contact (Definition 7.1). With the above theorem, the problem of finding the optimal grasp-set with independent regions of contact is equivalent to the problem of fitting a two-sided cone $C^{\times}(I,\ \mathcal{C})$, such that $C^{\times}(I,\ \mathcal{C})$ cuts the parallelograms Π_{12} and Π_{34} into two isosceles Π'_{12} and Π'_{34} with largest minimum side.

Notice that as we translate one of the edges of the cone $C^{\times}(I,\ \mathcal{C})$, the parallelograms Π'_{12} and Π'_{34} vary monotonically in opposite directions. This monotonicity allows us to design a constant time algorithm for finding the the best tradeoff position of I, or the optimal grasp-set. The algorithm is similar to Algorithm 7.1, and the details are skipped. It is more interesting to look at a high level description of the cone-fitting problem.

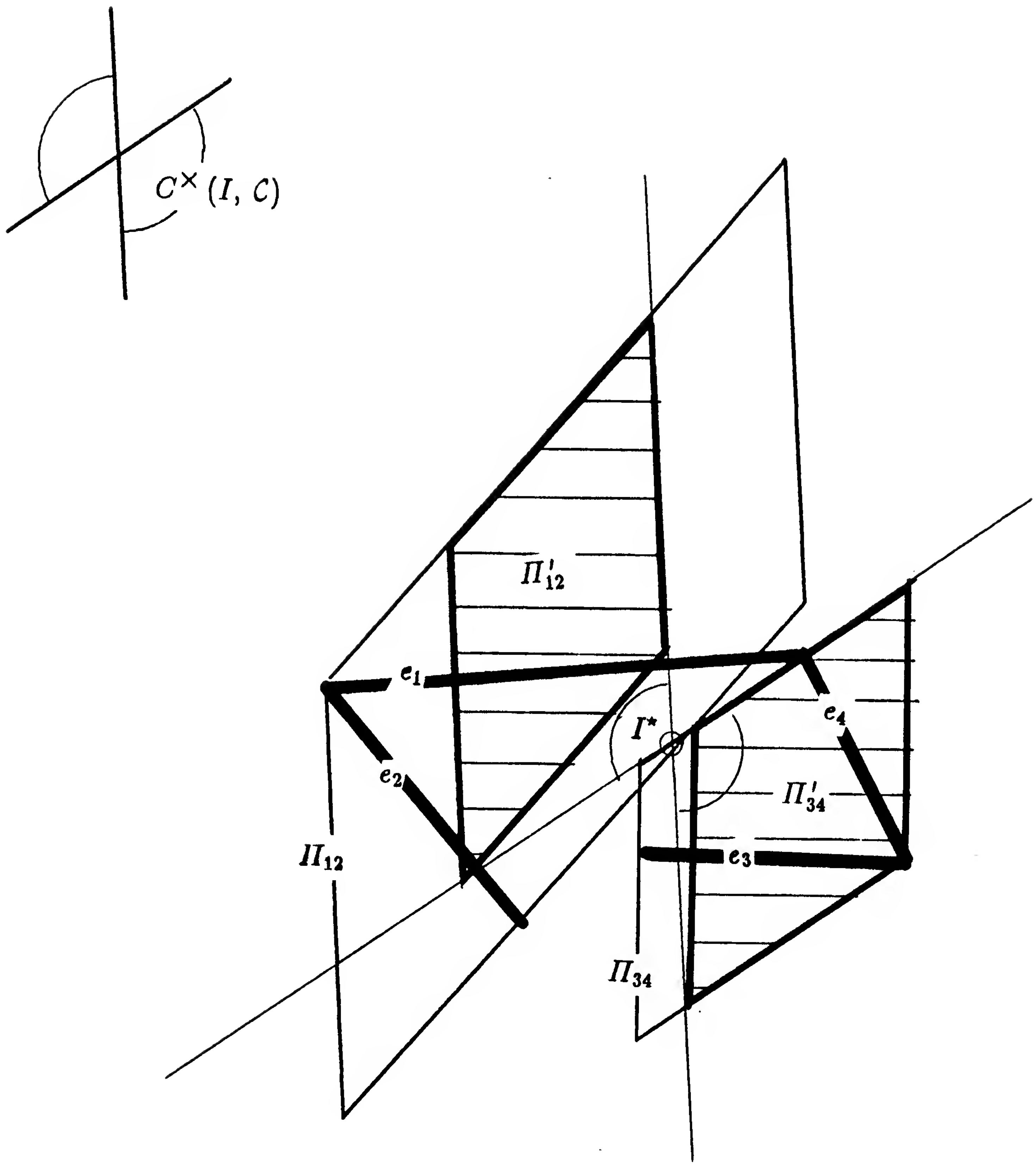


Figure 13: Finding the optimal set of grasps on four edges.

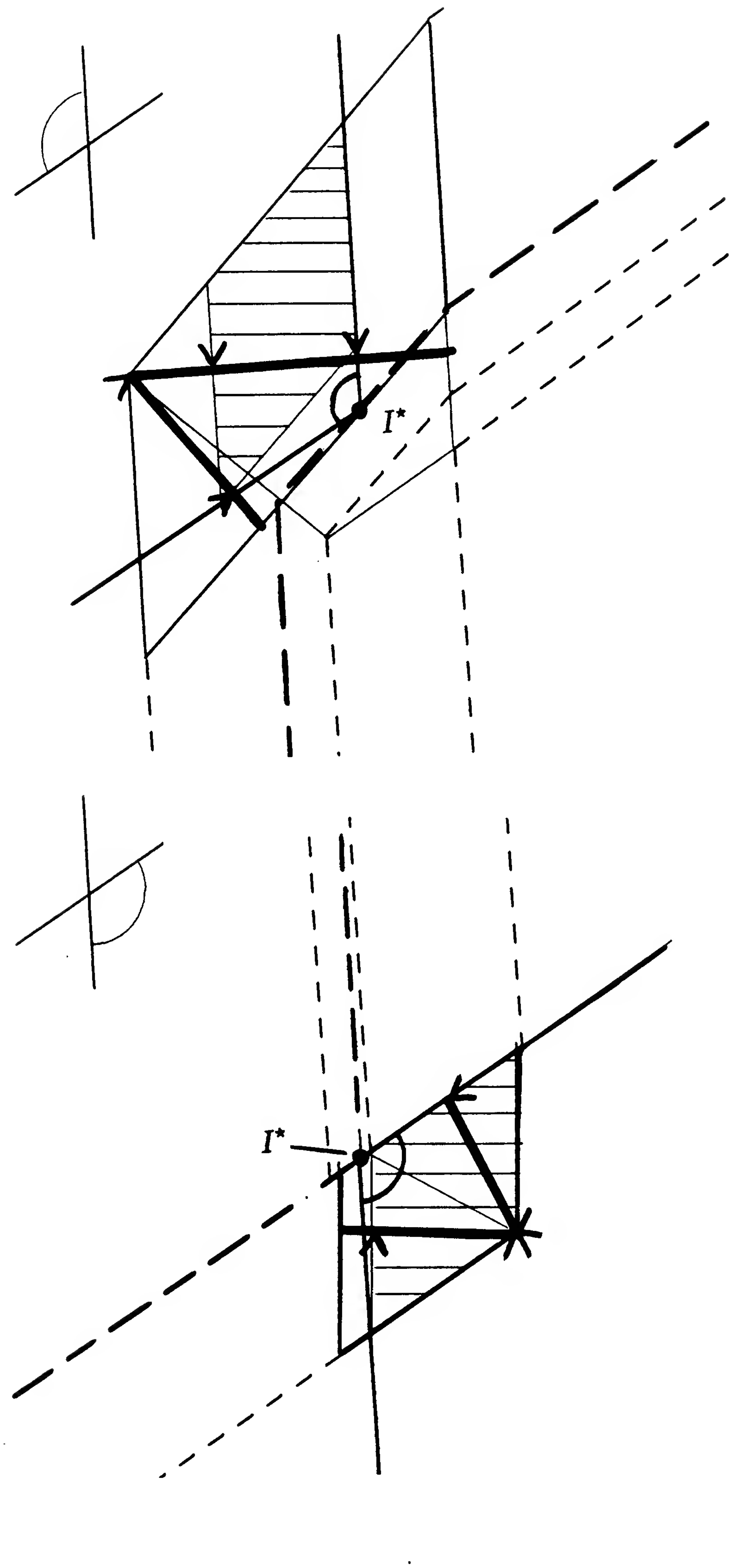


Figure 14: Search for the optimum vertex of the two-sided cone.

We partition the plane into regions depending on how the two-sided cone cuts the parallelogram Π_{12}, or how the contact segments overlap the two edges e_1 and e_2. In each of these regions, the loci of vertex I whose cone restricts the parallelogram Π_{12} into smaller parallelograms of constant area form parallel lines shown by dashed lines in Figure 14. We find similar regions and loci for parallelogram Π_{34}. The problem then reduces to an intersection of these two sets of loci, and a search for the best intersection point, or optimum vertex I. Finally, from this optimum vertex I, we deduce the two restricted parallelograms Π'_{12}, Π'_{34} that are largest and independent from each other. We back-project parallelogram Π'_{12}, (resp. Π'_{34}), to find the regions of contact r_1, r_2, (resp. r_3, r_4), on edges e_1, e_2, (resp. e_3, e_4).

The algorithm sketched above lumps edge e_1 with edge e_2, as one pair, and edge e_3 with edge e_4 as the other pair. One may wonder if a different pairing will give a different grasp-set. We have implemented the algorithm sketched as above and confirmed that the three best grasp sets found for $[(e_1,e_2)|(e_3,e_4)]$, $[(e_1,e_3)|(e_2,e_4)]$, $[(e_1,e_4)|(e_2,e_3)]$ give exactly the same set of contact segments on the four edges. The fact that the algorithm does not depend on the 2-2 paring between the edges is expected from Corollary 5.1. This fact reconfirms our earlier claim that the grasp-set specified by the parallelograms Π_{12}, Π_{34}, and the counter-overlapping sector $C_{12} \bigcap -C_{34}$ completely describes the set of grasps on the four edges e_1, e_2, e_3, e_4. This description is independent from the 2-2 pairing between the four edges.

Complexity 7.2 *Let B be the object grasped with four frictionless point contacts:*

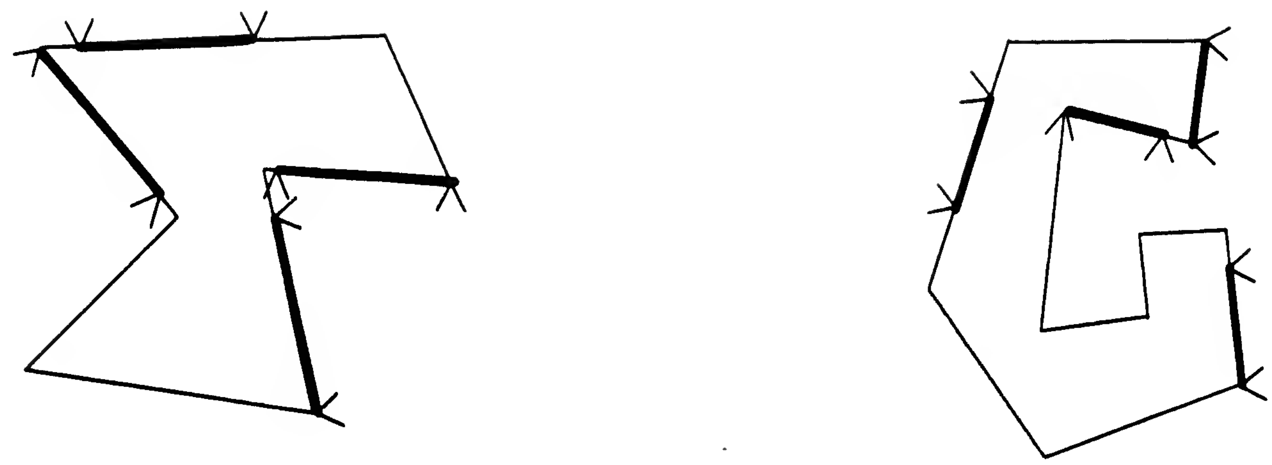
- *Finding the optimal set of grasps with independent regions of contact on four edges costs constant time.*
- *Finding the optimal 4-tuple of edges and its corresponding grasp-set costs $O|edges(B)|^4$. The four regions of contact found have the largest minimum length.*

8 Conclusion

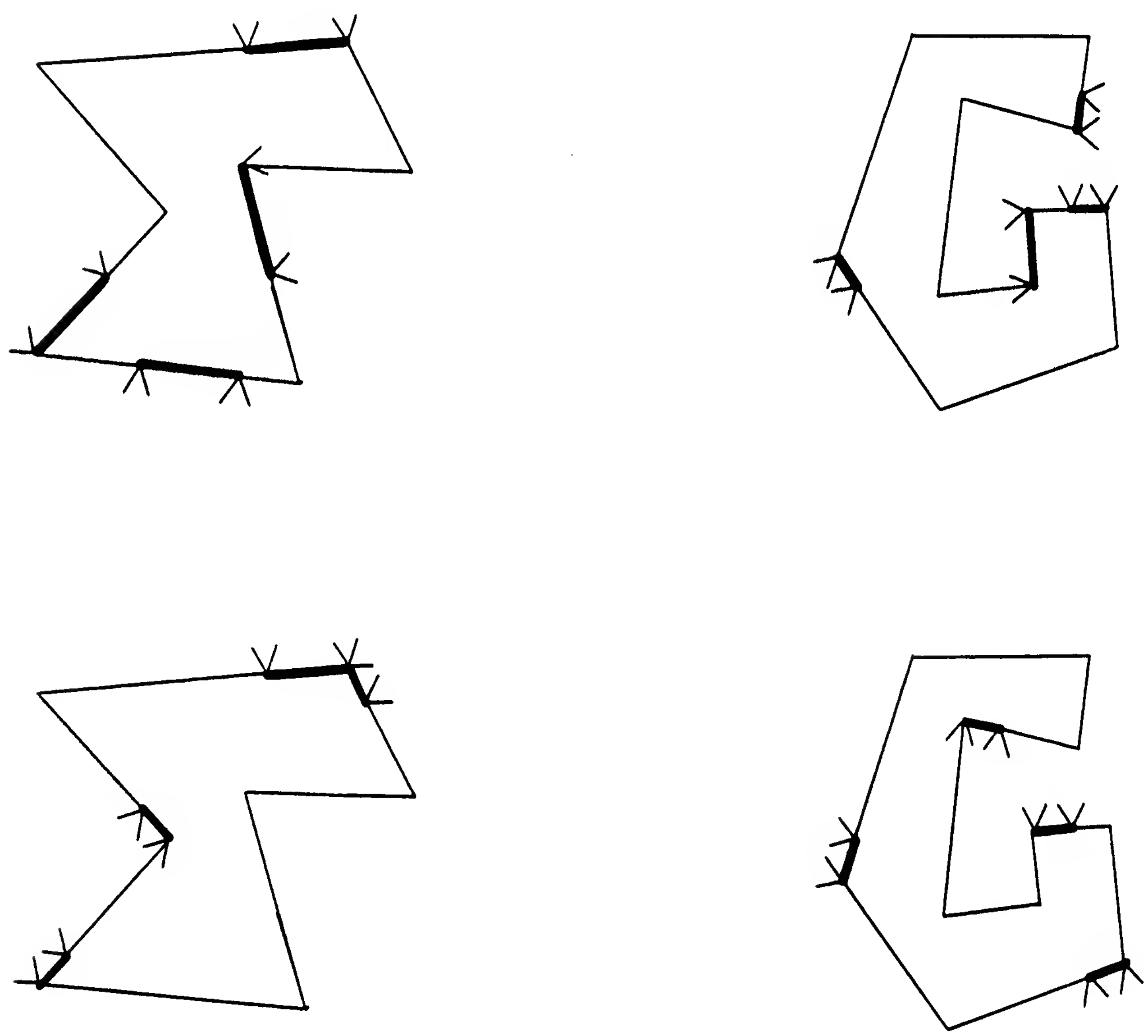
8.1 Performance

The synthesis of planar grasps with force closure has been fully implemented in Zeta Lisp, compiled and run on a Symbolics Lisp Machine. For the examples in Figure 1, the optimal set of grasps with independent regions of contact on two edges can be computed in 1/20 seconds. For four point contacts on three or four edges, this set of grasps can be computed in 1/4 seconds.

The minimum length of the independent regions of contact is a good measure of the tolerance of a grasp against inaccuracy in finger positioning. Finding the 4-tuple of grasping edges that has the largest minimum region of contact requires enumerating the maximal grasp-sets for all 4-tuples of edges. For a typical blob with 6 edges, this enumeration can take 10 seconds.



Grasps with maximum grasp resistance.



Grasps with minimum grasp resistance.

Figure 15: Grasps ranked by their resistance against unknown disturbance.

We can rank the grasp-sets based on some other criteria such as grasp resistance. Grasp resistance is defined as the work of the fingers in resisting unknown disturbance of the grasped object. It can be the sum of all individual work of each finger in the hand, or the maximum work of all fingers. We choose to take the maximum work of all individual fingers to penalize grasps that have one finger pressing very hard on the object while all the others just lightly touch the object. In other words, we want to minimize the maximal contact force applied on the object.

Since any motion of the object can be uniquely decomposed into a translation and a rotation about the origin, the grasp resistance against an arbitrary finite disturbance of the object can be split into the sum of a grasp resistance against arbitrary translation and and a grasp resistance against arbitrary rotation about the origin. The two partial grasp resistances must correspond to the same finger or the same contact, and we must compute the sum for each contact, and pick the maximum sum.

For simplicity, the current implementation computes the two grasp resistances independently from each other. In other words, for each contact, we compute the grasp resistance against rotation, then pick the maximum. We do the same for grasp resistance against translation, and sum these two. The resulting sum is only an approximation to the real grasp resistance. [4] We rank the grasp-sets in decreasing order of this approximate grasp resistance. Figure 15 shows the grasp-sets with best, medium, and worst grasp resistance. The disturbance of the object is an arbitrary translation of 1 centimeter and a rotation of 15 degrees.

8.2 Extensions

The current grasp synthesis can be extended in many directions:

• Applying other constraints such as the work space of the hand, or the shape of the hand and fingers. We want to make sure the grasp is feasible, or the fingers can wrap around and make contact with the object at the desired points. This problem can be formulated as a collision avoidance problem (Lozano-Perez 1983) by computing the CO of the grasped object in the configuration space of the fingers. We check to see if there exists a configuration of the hand and fingers that is outside of this CO-obstacle. Although general and mathematically complete, this scheme can be computationally very expensive because of the high dimension of the configuration space.

The current synthesis uses a very simple hand model with fingers like chop sticks perpendicular to the grasping plane. The finger tips can be anywhere inside a circular workspace.

• Adding other optimality criteria such as stability, resistance against the effect of gravity, or the effect of moment of inertia, etc... Just as we find restricted grasp-sets with independent regions of contact, we can search for the most stable grasp,

[4] Grasp resistance is computed only for the representative grasp at the mid points of the segments of contact.

assuming each finger is modeled as a spring (Hanafusa and Asada 1977, Baker et al. 1985, Nguyen 1985). (Nguyen 1985) shows that we can synthesize a set of virtual springs such that a force closure grasp is stable. Each finger is a virtual spring, and the contact is point contact without friction.

• The current synthesis can be easily extended to handle redundant contacts, or other types of contacts such as edge contacts and soft finger contacts. A more valuable extension is to synthesize force closure grasps of 3D objects. Interesting primitive configurations are grasps with two soft finger contacts, grasps with three point contacts with friction, or grasps with seven frictionless point contacts, etc... We are implementing the synthesis of 3D grasps. Results will be reported in (Nguyen 1986).

Acknowledgments

I am greatly indebted to Tomás Lozano-Pérez for his constant encouragement and for many helpful discussions. As always, he carefully reads a draft of this paper and and makes many comments which substantially improve the clarity of the paper.

I also would like to thank Kenneth Salisbury, Sundar Narasimhan, Alain Lanusse, Bruce Donald, James Mildrew for sharing my enthusiasm over new results and ideas, and for reading drafts of this paper.

I would like to thank the System Development Foundation for giving me a fellowship. Last but not least, my gratitude goes to my parents and to my fiancée for their support and love throughout this research.

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